PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On Electromagnetic Effects in the Theory of Shells and Plates

A. E. Green and P. M. Naghdi

Phil. Trans. R. Soc. Lond. A 1983 **309**, 559-610
doi: 10.1098/rsta.1983.0058

Phil. Trans. R. Soc. Lond. A **309**, 559–610 (1983) [559]
Printed in Great Britain

ON ELECTROMAGNETIC EFFECTS IN THE THEORY OF SHELLS AND PLATES

By A. E. Green†, F.R.S., and P. M. Naghdi‡
† *Mathematical Institute, University of Oxford*, 24–29 *St Giles, Oxford OX*1 3*LB, U.K.*
‡ *Department of Mechanical Engineering, University of California, Berkeley, California* 94720, *U.S.A.*

(*Received* 19 *August* 1982)

CONTENTS

This paper is concerned with the nonlinear and linear thermomechanical theories of deformable shell-like bodies in which account is taken of electromagnetic effects. The development is made by a direct approach with use of the two-dimensional theory of directed media called *Cosserat surfaces*. The first part of the paper deals with the formulation of appropriate nonlinear equations governing the motion of shell-like bodies in the presence of electromagnetic and thermal effects, as well as a general discussion of appropriate constitutive equations and symmetry restrictions. In the second part of the paper, attention is confined to special or more restrictive nonlinear and linear theories of shells including, for example, the nonlinear membrane theory, a restricted nonlinear bending theory (corresponding to the classical Kirchhoff–Love theory of shells) and a plate theory, all in the presence of electromagnetic effects. Finally, in the third part of the paper, attention is confined to specific topics, e.g. piezoelectricity in elastic plates and electromagnetic effects in a non-conducting plate, and a demonstration of the relevance and the applicability of the present direct formulation of a theory of electromagnetism for shell-like bodies.

1. Introduction

Electrodynamics of continuous media is a subject of considerable importance, with applications to both solids and fluids. As with continuum thermomechanics, when electromagnetic effects are absent, considerable difficulties occur when applying the three-dimensional theory to bodies with particular geometrical features such as plates and shells. Usually, some procedure is introduced to reduce the theory to two-dimensional form. When electromagnetic effects are present, no general theory seems to be available for shells or plates, although much work has been done for special problems such as the isothermal linear piezoelastic theory of plates. With use of a *direct* approach based on a two-dimensional continuum model known as *Cosserat* (or *directed*) *surfaces*, this paper is concerned with the nonlinear and linear theories of deformable shell-like bodies in which full account is taken of both electromagnetic and thermal effects. The two-dimensional continuum model, designated $\mathscr{C}_P$, comprises a material surface $\mathscr{S}$ embedded in a Euclidean 3-space together with $P (P = 1, 2, \ldots, N)$ deformable vector fields – called directors – attached to every point of the material surface. The directors, which are not necessarily along the unit normals to $\mathscr{C}_P$, have in particular the property that they remain unaltered in length under superposed rigid body motions.

In the absence of the directors, the two-dimensional continuum model is merely a two-dimensional material surface $\mathscr{S}$ appropriate for the construction by direct approach of the membrane theory of shells. With $P = 1$, the directed medium $\mathscr{C}_1 = \mathscr{C}$ consisting of the material surface $\mathscr{S}$ and a single deformable director is the simplest model for the construction of a general bending theory for thin shells and plates. The details of the basic theory of a Cosserat surface with a single director were given previously by Green *et al.* (1965) and by Naghdi (1972), where additional references predating 1972 can be found. The developments just referred to are made in the context of a thermomechanical theory of shells but allow for temperature changes only along some reference surface, such as the middle surface, of the (three-dimensional) shell-like body. The hierarchical theory of Cosserat surfaces, namely those comprising a material surface with $P\ (\geqslant 1)$ directors, was included in a paper by Green & Naghdi (1976), who subsequently also enlarged the scope of thermal effects by incorporating into the basic theory the effect of temperature changes along the shell thickness (Green & Naghdi 1979). This development was achieved by means of an approach to thermomechanics in the three-dimensional theory introduced earlier (Green & Naghdi 1977), which provides a natural way of introducing two (or more) temperature fields at each material point of the surface $\mathscr{S}$ of $\mathscr{C}_P$. Additional background information on purely mechanical or thermomechanical theories of shells can be found in the references already cited and in a recent paper of Naghdi (1982).

We now turn to some background information concerning electromagnetic effects. At present, a (three-dimensional) theory of deformable media in the presence of electromagnetic effects may be developed at a number of levels of generality. One approach is based on a mixture theory in which the thermomechanical continuum is one constituent interacting with electric particle continua which, in turn, are acted upon by forces due to the electromagnetic fields. Another approach, adopted here, is to ignore details of the electric particle continua and consider only a single phase theory in which the continuum is acted upon directly by electromagnetic forces. We use an approximate non-relativistic theory in which the balance equations and also Maxwell's equations are invariant under a Galilean transformation of the form

$$\boldsymbol{r}^{*+} = \boldsymbol{Q}\boldsymbol{r}^* + \boldsymbol{\lambda}t, \quad t^+ = t$$

where $\boldsymbol{r}^*$ is the position vector of a material point of the body, t denotes time, the use of a plus sign (as in $\boldsymbol{r}^{*+}$ and t^+) refers to the corresponding quantities as a consequence of superposed rigid body motions, $\boldsymbol{\lambda}$ is a constant vector and $\boldsymbol{Q}$ is a constant orthogonal tensor. In addition, constitutive equations are to be unaltered by a constant superposed rigid body velocity and a constant superposed rigid body rotation. With these limitations, many authors have derived values for the three-dimensional electromagnetic force $\boldsymbol{f}^*_{\mathrm{e}}$, the electromagnetic couple $\boldsymbol{c}^*_{\mathrm{e}}$ and the rate of supply of electromagnetic energy w^*, and have discussed various constitutive relations. A survey of these various theories on the subject, together with extensive references, are given in a monograph by Hutter & van de Ven (1978). The survey of electromechanical interaction effects is presented in the form of five models, which the authors (Hutter & van de Ven 1978) refer to as the two-dipole models (i) and (ii), the Maxwell–Minkowski model, the statistical model and the Lorentz model. Although these models yield different values for the electromagnetic force, couple and rate of supply of electromagnetic energy, Hutter & van de Ven (1978) show that for a certain class of constitutive equations all theories are equivalent within the non-relativistic approximation. For our purpose, we select here three-dimensional values for $\boldsymbol{f}^*_{\mathrm{e}}$, $\boldsymbol{c}^*_{\mathrm{e}}$ and w^* that are a slight modification of the Maxwell–Minkowski model discussed by Hutter & van de Ven (1978). As will become evident, the theory for shells based on the Cosserat surfaces $\mathscr{C}_P$ will reflect the properties of this model, whose main equations are summarized in Appendixes A, B.

The linear theory of plates and its application to anisotropic piezoelectric plates has received much attention from a number of authors. Among these we mention Tiersten & Mindlin (1962) and Tiersten (1969), who cite additional references. These authors derived their equations from the three-dimensional equations of linear piezoelasticity with the help of expansion methods due to Cauchy and Poisson and the variational method of Kirchhoff, together with the use of correction factors involving the thickness-shear strains. The expansions are in powers of the thickness coordinate z of the plate. More recently, Bugdayci & Bogy (1981) used expansions in terms of trigonometric functions of z since these were more suitable for the particular boundary conditions they used on the major surfaces $z = \pm\frac{1}{2}h$ of the plate. The linearized version of our developments when specialized to a plate, can be used to accommodate any of the surface conditions used by the foregoing authors and with as much generality as that in the paper by Bugdayci & Bogy (1981).

Specifically, the content of the paper is as follows. First, with reference to Cosserat surfaces $\mathscr{C}_P$, in § 2 the basic thermomechanical theory with extensions to electromagnetic effects is summarized and the consequences of the conservation laws in direct (coordinate free) notation are recorded in both spatial and material (or referential) forms. This is followed in § 3 by appropriate electromagnetic balance equations for a moving shell-like body. These balance laws are analogues of corresponding conservation laws in the three-dimensional theory, which are summarized and discussed in Appendixes A and B. In § 4, we consider constitutive equations for magnetic, polarized thermoelastic Cosserat surfaces $\mathscr{C}_P$. Section 5 contains some discussion concerning restrictions placed on the specific Helmholtz free energy function by symmetry properties, particularly the separation of geometrical symmetries in the shell-like body from material symmetries.

In the next four sections (§§ 6–9), we consider special cases of the general theory such as a restricted nonlinear theory of shells (corresponding to the classical Kirchhoff–Love theory), the membrane theory, and linear plate theory, all in the presence of electromagnetic effects. The linearized theory of plates, based on a Cosserat surface with material surface $\mathscr{S}$ taken to be a plane surface in the reference configuration of $\mathscr{C}$, is dealt with in some detail in § 6 for an anisotropic plate. The constitutive coefficients in this development are determined by an appeal to, and comparison with, certain features of the Helmholtz free energy and related Gibbs free energy functions in the three-dimensional theory. The details of these functions are discussed in Appendixes C and D. A partially restricted linear theory of shells in which the effect of transverse shear deformation is retained is discussed in § 7; and again the identification of constitutive coefficients, with the help of results in Appendixes C and D, is indicated in a manner similar to the procedure used in § 6. The next two sections (§§ 8, 9) deal, respectively, with a restricted nonlinear theory of shells and the nonlinear membrane theory, including the linearized results. In the development of both §§ 8 and 9, full allowance is made for electromagnetic effects.

The remainder of the paper is devoted to further special cases and applications of earlier developments. In § 10 the partially restricted linear theory of § 7 is used to discuss electromagnetic effects in a non-conducting plate whose major surfaces are also heat insulated. Again with use of the partially restricted linear theory of § 7, the piezoelectric crystal plates are briefly examined in § 11 and an alternative representation of the plate theory is discussed in § 12, where application to an elastic wave guide is also discussed. We conclude the present paper with two further applications, which are discussed in §§ 13 and 14 for a circular cylindrical membrane and for a wave guide regarded as a rigid shell.

2. Summary of thermomechanical theory with extensions to electromagnetic effects

We summarize in this section the main kinematics and the basic equations of the thermomechanical theory of Cosserat surfaces $\mathscr{C}_P$, with extensions to include electromagnetic effects, and refer to Naghdi (1972, 1982) and Green & Naghdi (1976, 1979) for details and additional references. Let the particles of the material surface of $\mathscr{C}_P$ (with P directors, $P = 1, 2, 3, \ldots, N$) be identified with a system of convected coordinates θ^α $(\alpha = 1, 2)$ and let the surface of $\mathscr{C}_P$ in the present configuration at time t, hereafter referred to as $\mathscr{s}$, occupy a two-dimensional region of space $\mathscr{R}$ bounded by a closed curve $\partial\mathscr{R}$. Similarly, in the present configuration, an arbitrary part of the material surface of $\mathscr{C}_P$ occupies a portion of the two-dimensional region $\mathscr{R}$, which we denote by $\mathscr{P}$ $(\subseteq \mathscr{R})$ bounded by a closed curve $\partial\mathscr{P}$. Let $\boldsymbol{r}$ and $\boldsymbol{d}_N$ $(N = 1, 2, \ldots, P)$ – each a function of θ^α and t – denote, respectively, the position vectors of a typical point of $\mathscr{s}$ relative to a fixed origin and the directors at $\boldsymbol{r}$. We denote the covariant and contravariant base vectors on $\mathscr{s}$ by $\boldsymbol{a}_\alpha$ and $\boldsymbol{a}^\alpha$, the unit normal to $\mathscr{s}$ by $\boldsymbol{a}_3$ $(= \boldsymbol{a}^3)$, and the covariant and contravariant metric tensors on $\mathscr{s}$ by $a_{\alpha\beta}$ and $a^{\alpha\beta}$. A motion of the Cosserat surfaces $\mathscr{C}_P$ is defined by vector-valued functions, which assign position $\boldsymbol{r}$ and directors $\boldsymbol{d}_N$ to each particle of the material surface of $\mathscr{C}_P$ at each instant of time, i.e.

$$\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha, t), \quad \boldsymbol{d}_N = \boldsymbol{d}_N(\theta^\alpha, t). \tag{2.1}$$

Also,
$$\left.\begin{aligned} &\boldsymbol{a}_\alpha = \boldsymbol{a}_\alpha(\theta^\beta, t) = \partial \boldsymbol{r}/\partial\theta^\alpha, \quad a_{\alpha\beta} = \boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta, \quad \boldsymbol{a}^\alpha\cdot\boldsymbol{a}_\beta = \delta^\alpha_\beta, \quad a^{\alpha\beta} = \boldsymbol{a}^\alpha\cdot\boldsymbol{a}_\beta, \\ &a^{\frac{1}{2}}\boldsymbol{a}_3 = \boldsymbol{a}_1\times\boldsymbol{a}_2, \quad a = \det(a_{\alpha\beta}), \quad a^{\frac{1}{2}} = [\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3] > 0, \end{aligned}\right\} \tag{2.2}$$

where δ^α_β is the Kronecker delta. The velocity vector $\boldsymbol{v}$ and the director velocity vectors $\boldsymbol{w}_N$ are defined by
$$\boldsymbol{v} = \dot{\boldsymbol{r}}, \quad \boldsymbol{w}_N = \dot{\boldsymbol{d}}_N, \tag{2.3}$$

where a dot denotes differentiation with respect to t, with θ^α held fixed. Throughout this paper we use standard vector and tensor notation. Greek indices take the values 1, 2, and the usual summation convention over a Greek superscript and subscript is followed.

Consider now a reference configuration, which we take to be the initial configuration, of the Cosserat surfaces $\mathscr{C}_P$. Let the material surface of $\mathscr{C}_P$ in this configuration be referred to by $\mathscr{S}$ with $\boldsymbol{R}$ as its position vector; let $\boldsymbol{A}_\alpha$, $\boldsymbol{A}^\alpha$ denote, respectively, the covariant and contravariant base vectors along the θ^α-curves on $\mathscr{S}$, $\boldsymbol{A}_3$ $(=\boldsymbol{A}^3)$ the unit normal to $\mathscr{S}$, and $A_{\alpha\beta}$, $A^{\alpha\beta}$ the covariant and contravariant metric tensors on $\mathscr{S}$, respectively; and let $\boldsymbol{D}_N$ be the reference directors at $\boldsymbol{R}$. Then
$$\left.\begin{aligned} &\boldsymbol{R} = \boldsymbol{R}(\theta^\alpha) = \boldsymbol{r}(\theta^\alpha, 0), \quad \boldsymbol{D}_N = \boldsymbol{D}_N(\theta^\alpha) = \boldsymbol{d}_N(\theta^\alpha, 0), \\ &\boldsymbol{A}_\alpha = \partial\boldsymbol{R}/\partial\theta^\alpha, \quad A_{\alpha\beta} = \boldsymbol{A}_\alpha\cdot\boldsymbol{A}_\beta, \quad \boldsymbol{A}^\alpha\cdot\boldsymbol{A}_\beta = \delta^\alpha_\beta, \quad A^{\alpha\beta} = \boldsymbol{A}^\alpha\cdot\boldsymbol{A}^\beta, \\ &A^{\frac{1}{2}}\boldsymbol{A}_3 = \boldsymbol{A}_1\times\boldsymbol{A}_2, \quad A = \det(A_{\alpha\beta}), \quad A^{\frac{1}{2}} = [\boldsymbol{A}_1\boldsymbol{A}_2\boldsymbol{A}_3] > 0. \end{aligned}\right\} \tag{2.4}$$

We also define a set of linearly independent vectors $\boldsymbol{d}_{Ni}$ and their reciprocal vectors $\boldsymbol{d}^i_N$ $(i = 1, 2, 3)$, as well as their corresponding values $\boldsymbol{D}_{Ni}$ and $\boldsymbol{D}^i_N$ in the reference configuration, by the formulae
$$\left.\begin{aligned} &\boldsymbol{d}_{N\alpha} = \boldsymbol{a}_\alpha, \quad \boldsymbol{d}_{N3} = \boldsymbol{d}_N, \quad \boldsymbol{d}^i_N\cdot\boldsymbol{d}_{Nj} = \delta^i_j, \quad \boldsymbol{d}_N = d_{Ni}\,\boldsymbol{a}^i = d_N{}^i\boldsymbol{a}_i, \\ &\boldsymbol{D}_{N\alpha} = \boldsymbol{A}_\alpha, \quad \boldsymbol{D}_{N3} = \boldsymbol{D}_N, \quad \boldsymbol{D}^i_N\cdot\boldsymbol{D}_{Nj} = \delta^i_j, \end{aligned}\right\} \tag{2.5}$$

where δ^i_j is the Kronecker delta in 3-space. There is no summation over repeated values of N. For some purposes it is convenient to use a direct (coordinate-free) notation. To this end, with some changes but in a manner similar to that of Naghdi (1977, 1982), we introduce a measure of deformation gradient tensor $\boldsymbol{F}$, director gradient tensors $\boldsymbol{G}_N$ and ${}_{\mathrm{R}}\boldsymbol{G}_N$, the velocity gradient $\boldsymbol{L}$ and the director velocity gradient $\boldsymbol{L}_N$ by:
$$\left.\begin{aligned} &\boldsymbol{F} = \boldsymbol{a}_i\otimes\boldsymbol{A}^i, \quad \boldsymbol{F}_N = \boldsymbol{d}_{N\alpha}\otimes\boldsymbol{D}^\alpha_N + \boldsymbol{d}_N\otimes\boldsymbol{D}^3_N, \quad {}_{\mathrm{R}}\boldsymbol{F} = \boldsymbol{A}_i\otimes\boldsymbol{A}^i, \\ &\boldsymbol{a}_i = \boldsymbol{F}\boldsymbol{A}_i, \quad \boldsymbol{d}_{N\alpha} = \boldsymbol{F}_N\boldsymbol{D}_{N\alpha}, \quad \boldsymbol{d}_N = \boldsymbol{F}_N\boldsymbol{D}_N, \quad \det\boldsymbol{F} = a^{\frac{1}{2}}/A^{\frac{1}{2}}, \\ &\boldsymbol{G}_N = \boldsymbol{d}_{N,\alpha}\otimes\boldsymbol{D}^\alpha_N + \boldsymbol{d}_N\otimes\boldsymbol{D}^3_N, \quad \boldsymbol{d}_N = \boldsymbol{G}_N\boldsymbol{D}_N, \\ &\boldsymbol{d}_{N,\alpha} = \boldsymbol{G}_N\boldsymbol{D}_{N\alpha} = d_{Ni\alpha}\,\boldsymbol{a}^i = d_N{}^i{}_\alpha\,\boldsymbol{a}_i, \\ &{}_t\boldsymbol{G}_N = \boldsymbol{d}_{N,\alpha}\otimes\boldsymbol{d}^\alpha_N + \boldsymbol{d}_N\otimes\boldsymbol{d}^3_N, \quad {}_{\mathrm{R}}\boldsymbol{G}_N = \boldsymbol{D}_{N,\alpha}\otimes\boldsymbol{D}^\alpha_N + \boldsymbol{D}_N\otimes\boldsymbol{D}^3_N, \\ &\boldsymbol{L} = \dot{\boldsymbol{a}}_i\otimes\boldsymbol{a}^i, \quad \dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}, \\ &\boldsymbol{L}_N = \boldsymbol{w}_{N,\alpha}\otimes\boldsymbol{d}^\alpha_N + \boldsymbol{w}_N\otimes\boldsymbol{d}^3_N, \quad \dot{\boldsymbol{G}}_N = \boldsymbol{L}_N\boldsymbol{F}_N. \end{aligned}\right\} \tag{2.6}$$

Here a comma denotes partial differentiation with respect to θ^α, the symbol $\otimes$ denotes tensor product and there is no summation over repeated indices N.

In what follows, for convenience and completeness, we express the basic equations of the theory in both spatial and material (or referential) forms. We begin with the spatial form of the conservation laws for mass momentum, director momentum and moment of momenta, which may be stated as†

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\rho\,\mathrm{d}\sigma = 0, \tag{2.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\rho\left(\boldsymbol{v}+\sum_{M=1}^{P}y^{M0}\boldsymbol{w}_M\right)\mathrm{d}\sigma = \int_{\mathscr{P}}\rho(\boldsymbol{f}+\boldsymbol{f}_\mathrm{e})\,\mathrm{d}\sigma+\int_{\partial\mathscr{P}}\boldsymbol{n}\,\mathrm{d}s, \tag{2.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\rho\left(y^{N0}\boldsymbol{v}+\sum_{N=1}^{P}y^{NM}\boldsymbol{w}_M\right)\mathrm{d}\sigma = \int_{\mathscr{P}}\{\rho(\boldsymbol{l}^N+\boldsymbol{l}^N_\mathrm{e})-\boldsymbol{k}^N\}\,\mathrm{d}\sigma+\int_{\partial\mathscr{P}}\boldsymbol{m}^N\,\mathrm{d}s, \tag{2.9}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}&\left\{\boldsymbol{r}\times\left(\boldsymbol{v}+\sum_{M=1}^{P}y^{M0}\boldsymbol{w}_M\right)+\sum_{N=1}^{P}\boldsymbol{d}_N\times\left(y^{N0}\boldsymbol{v}+\sum_{M=1}^{P}y^{NM}\boldsymbol{w}_M\right)\right\}\rho\,\mathrm{d}\sigma\\ &=\int_{\mathscr{P}}\left\{\boldsymbol{r}\times(\boldsymbol{f}+\boldsymbol{f}_\mathrm{e})+\sum_{N=1}^{P}\boldsymbol{d}_N\times(\boldsymbol{l}^N+\boldsymbol{l}^N_\mathrm{e})+\boldsymbol{c}_\mathrm{e}\right\}\rho\,\mathrm{d}\sigma+\int_{\partial\mathscr{P}}\left(\boldsymbol{r}\times\boldsymbol{n}+\sum_{N=1}^{P}\boldsymbol{d}_N\times\boldsymbol{m}^N\right)\mathrm{d}s.\end{aligned} \tag{2.10}$$

In (2.7)–(2.10), $\rho = \rho(\theta^\gamma, t)$ is the mass per unit area of $\mathscr{s}$, $\mathrm{d}\sigma$ is the element of area and $\mathrm{d}s$ is the line element on $\mathscr{s}$, y^{N0} and y^{NM} are the inertia coefficients, which are functions of θ^γ and independent of time t, $\boldsymbol{n} = \boldsymbol{n}(\theta^\gamma, t; \boldsymbol{\nu})$ is the force vector, $\boldsymbol{m}^N = \boldsymbol{m}^N(\theta^\gamma, t; \boldsymbol{\nu})$ are the director force vectors, $\boldsymbol{k}^N$ are the internal director forces per unit area of $\mathscr{s}$ and

$$\boldsymbol{\nu} = \nu_\alpha \boldsymbol{a}^\alpha = \nu^\alpha \boldsymbol{a}_\alpha \tag{2.11}$$

is the outward unit normal to $\partial\mathscr{P}$. Also, $\boldsymbol{f}$ is the assigned force vector, $\boldsymbol{l}^N$ are the assigned director force vectors and $\boldsymbol{f}_\mathrm{e}$, $\boldsymbol{l}^N_\mathrm{e}$, $\boldsymbol{c}_\mathrm{e}$ are, respectively, the force vector, the director force vectors and the (axial) couple vector due to the electromagnetic fields, all per unit mass. The assigned field $\boldsymbol{f}$ may be regarded as representing the combined effect of (i) the stress vector on the major surfaces of the shell, denoted by $\boldsymbol{f}_\mathrm{c}$, for example that due to the ambient pressure of the surrounding medium, and (ii) an integrated contribution arising from the three-dimensional body force denoted by $\boldsymbol{f}_\mathrm{b}$, for example that due to gravity. A parallel statement holds for the assigned fields $\boldsymbol{l}^N$. Therefore we may write

$$\boldsymbol{f} = \boldsymbol{f}_\mathrm{b}+\boldsymbol{f}_\mathrm{c}, \quad \boldsymbol{l}^N = \boldsymbol{l}^N_\mathrm{b}+\boldsymbol{l}^N_\mathrm{c}. \tag{2.12}$$

We could, of course, incorporate $\boldsymbol{f}_\mathrm{e}$ and $\boldsymbol{l}^N_\mathrm{e}$ as parts of $\boldsymbol{f}$ and $\boldsymbol{l}^N$, respectively; but, for our present purpose, it is more convenient to keep the electromagnetic effects $\boldsymbol{f}_\mathrm{e}$, $\boldsymbol{l}^N_\mathrm{e}$ separate from those of the purely mechanical effects represented by (2.12).

Using a direct (coordinate-free) notation (Naghdi 1982, § 8), the local field equations resulting from (2.7)–(2.10) can be expressed in the forms‡

$$\dot\rho+\rho\,\mathrm{div}_s\,\boldsymbol{v} = 0 \quad \text{or} \quad \rho a^{\frac{1}{2}} = \rho_\mathrm{R} A^{\frac{1}{2}}, \tag{2.13}$$

$$\rho\left(\dot{\boldsymbol{v}}+\sum_{M=1}^{P}y^{M0}\dot{\boldsymbol{w}}_M\right) = \rho(\boldsymbol{f}+\boldsymbol{f}_\mathrm{e})+\mathrm{div}_s\,\boldsymbol{N}, \tag{2.14}$$

$$\rho\left(y^{N0}\dot{\boldsymbol{v}}+\sum_{M=1}^{P}y^{NM}\dot{\boldsymbol{w}}_M\right) = \rho(\boldsymbol{l}^N+\boldsymbol{l}^N_\mathrm{e})-\boldsymbol{k}^N+\mathrm{div}_s\,\boldsymbol{M}^N, \tag{2.15}$$

$$\rho\boldsymbol{\Gamma}_\mathrm{e}+\boldsymbol{N}-\boldsymbol{N}^\mathrm{T}+\sum_{N=1}^{P}(\boldsymbol{K}^N-\boldsymbol{K}^{N\mathrm{T}}+\boldsymbol{M}^N{}_t\boldsymbol{G}^\mathrm{T}_N-{}_t\boldsymbol{G}_N\boldsymbol{M}^{N\mathrm{T}}) = \boldsymbol{0}, \tag{2.16}$$

† The notation for the contact force $\boldsymbol{n}$, the contact director force $\boldsymbol{m}$ and the surface director force $\boldsymbol{k}$ is the same as that in Naghdi (1982), but differs from Naghdi (1972), Green & Naghdi (1979) and most of the previous papers on the subject. In fact, the vector fields $\boldsymbol{n}$, $\boldsymbol{m}$, $\boldsymbol{k}$ of the present paper correspond, respectively, to $\boldsymbol{N}$, $\boldsymbol{M}$, $\boldsymbol{m}$ in Naghdi (1972), Green & Naghdi (1979) and most of the previous papers. Also the notation for the inertia coefficients y^{M0} and y^{MN}, which occur in (2.8)–(2.10), differs from the corresponding notation in previous papers.

‡ The second-order tensors $\boldsymbol{N}$, $\boldsymbol{M}$ in (2.14)–(2.16) and their tensor components $N^{i\alpha}$, $M^{i\alpha}$ in (2.17) are the same as those in Naghdi (1982) but are the transpose of the corresponding quantities in Naghdi (1977). The components $N^{i\alpha}$, $M^{i\alpha}$ were used in the paper of Green *et al.* (1965), in order to conform with the linear transformations $\boldsymbol{N\nu}$ and $\boldsymbol{M\nu}$; but their transpose, namely $N^{\alpha i}$, $M^{\alpha i}$, was adopted in subsequent papers, so that the notation would be in agreement with that of the classical shell theory. In this connection, see also Naghdi (1982, §8).

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

or

$$\rho \boldsymbol{c}_{\mathrm{e}} + \boldsymbol{a}_{\alpha} \times \boldsymbol{N}^{\alpha} + \sum_{N=1}^{P} (\boldsymbol{d}_N \times \boldsymbol{k}^N + \boldsymbol{d}_{N,\alpha} \times \boldsymbol{M}^{N\alpha}) = \boldsymbol{0}, \tag{2.16a}$$

where $\boldsymbol{\Gamma}_{\mathrm{e}} \boldsymbol{z} = \boldsymbol{c}_{\mathrm{e}} \times \boldsymbol{z}$ for every vector $\boldsymbol{z}$. Also, in (2.13)–(2.16), the second-order tensors $\boldsymbol{N}$, $\boldsymbol{M}^N$, $\boldsymbol{K}^N$ are defined by

$$\left.\begin{aligned} \boldsymbol{n} = \boldsymbol{N}\boldsymbol{v} = \boldsymbol{N}^{\alpha} v_{\alpha}, \quad \boldsymbol{N} = \boldsymbol{N}^{\alpha} \otimes \boldsymbol{a}_{\alpha}, \quad \boldsymbol{N}^{\alpha} = N^{i\alpha} \boldsymbol{a}_i, \\ \boldsymbol{m}^N = \boldsymbol{M}^N \boldsymbol{v} = \boldsymbol{M}^{N\alpha} v_{\alpha}, \quad \boldsymbol{M}^N = \boldsymbol{M}^{N\alpha} \otimes \boldsymbol{a}_{\alpha}, \quad \boldsymbol{M}^{N\alpha} = M^{Ni\alpha} \boldsymbol{a}_i, \\ \boldsymbol{k}^N = k^{Ni} \boldsymbol{a}_i = k_i^N \boldsymbol{a}^i, \quad \boldsymbol{K}^N = \boldsymbol{k}^N \otimes \boldsymbol{d}_N \end{aligned}\right\} \tag{2.17}$$

and

$$\left.\begin{aligned} \operatorname{div}_s \boldsymbol{v} &= \boldsymbol{v}_{,\alpha} \cdot \boldsymbol{a}^{\alpha}, \\ a^{\frac{1}{2}} \operatorname{div}_s \boldsymbol{N} &= (a^{\frac{1}{2}} \boldsymbol{N} \boldsymbol{a}^{\alpha})_{,\alpha} = (a^{\frac{1}{2}} \boldsymbol{N}^{\alpha})_{,\alpha}, \\ a^{\frac{1}{2}} \operatorname{div}_s \boldsymbol{M}^N &= (a^{\frac{1}{2}} \boldsymbol{M}^N \boldsymbol{a}^{\alpha})_{,\alpha} = (a^{\frac{1}{2}} \boldsymbol{M}^{N\alpha})_{,\alpha}. \end{aligned}\right\} \tag{2.18}$$

The balances of entropy and energy for every part of the material surface of $\mathscr{C}$ occupying a region $\mathscr{P}$ in the present configuration are (see Green & Naghdi 1979):

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{P}} \rho \eta \, \mathrm{d}\sigma = \int_{\mathscr{P}} \rho (s + \xi) \, \mathrm{d}\sigma - \int_{\partial\mathscr{P}} k \, \mathrm{d}s, \tag{2.19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{P}} \rho \eta_N \, \mathrm{d}\sigma = \int_{\mathscr{P}} \rho (s_N + \xi_N) \, \mathrm{d}\sigma - \int_{\partial\mathscr{P}} k_N \, \mathrm{d}s \quad (N = 1, 2, \ldots, K) \tag{2.20}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{P}} \left(\epsilon + \tfrac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v} + \sum_{N=1}^{P} y^{N0} \boldsymbol{v} \cdot \boldsymbol{w}_N + \tfrac{1}{2} \sum_{N=1}^{P} \sum_{M=1}^{P} y^{NM} \boldsymbol{w}_N \cdot \boldsymbol{w}_M \right) \rho \, \mathrm{d}\sigma \\ = \int_{\mathscr{P}} \left\{ r + \sum_{N=1}^{K} r_N + (\boldsymbol{f} + \boldsymbol{f}_{\mathrm{e}}) \cdot \boldsymbol{v} + \sum_{N=1}^{P} (\boldsymbol{l}^N + \boldsymbol{l}_{\mathrm{e}}^N) \cdot \boldsymbol{w}_N + \bar{w} \right\} \rho \, \mathrm{d}\sigma \\ + \int_{\partial\mathscr{P}} \left(\boldsymbol{n} \cdot \boldsymbol{v} + \sum_{N=1}^{P} \boldsymbol{m}^N \cdot \boldsymbol{w}_N - h - \sum_{N=1}^{K} h_N \right) \mathrm{d}s. \end{aligned} \tag{2.21}$$

In (2.19)–(2.21), η and η_N are the entropy densities, ϵ is the internal energy density, k and k_N are the entropy fluxes, h and h_N are the heat fluxes, ξ and ξ_N are the internal rates of production of entropy, s and s_N are the external rates of supply of entropy, r and r_N are the external rates of supply of heat, and

$$r = \theta s, \quad r_N = \theta_N s_N, \quad h = \theta k, \quad h_N = \theta_N k_N \quad (N = 1, 2, \ldots, K), \tag{2.22}$$

where† $\theta(>0)$ and θ_N represent the effects of temperature variations in a shell-like body. The surface temperature θ represents the absolute temperature in the reference surface of the shell-like body, while the scalars θ_N may be regarded as accounting for the temperature variations across the thickness of the shell. Also, the scalar quantity $\bar{w}$ on the right-hand side of (2.21) represents the combined effect of both the rate of work of the electromagnetic couple $\boldsymbol{c}_{\mathrm{e}}$ and the rate of supply of electromagnetic energy due to the magnetic field.

The local field equations that correspond to (2.19) to (2.21) are:

$$\left.\begin{aligned} \rho \dot{\eta} = \rho (s + \xi) - \operatorname{div}_s \boldsymbol{p}, \quad k = \boldsymbol{p} \cdot \boldsymbol{v}, \\ \rho \dot{\eta}_N = \rho (s_N + \xi_N) - \operatorname{div}_s \boldsymbol{p}_N, \quad k_N = \boldsymbol{p}_N \cdot \boldsymbol{v}, \end{aligned}\right\} \tag{2.23}$$

where

$$a^{\frac{1}{2}} \operatorname{div}_s \boldsymbol{p} = (a^{\frac{1}{2}} \boldsymbol{p} \cdot \boldsymbol{a}^{\alpha})_{,\alpha}, \quad a^{\frac{1}{2}} \operatorname{div}_s \boldsymbol{p}_N = (a^{\frac{1}{2}} \boldsymbol{p}_N \cdot \boldsymbol{a}^{\alpha})_{,\alpha}. \tag{2.24}$$

† No confusion should arise from the use of the symbol θ in the designation of the temperature fields as $\theta, \theta_1, \theta_2, \ldots, \theta_K$ and the notation $\theta^{\gamma} = (\theta^1, \theta^2)$ for the convected coordinates.

After elimination of external body forces and external rates of supply of heat with the help of (2.14), (2.15), (2.22) and (2.23), the field equation corresponding to the energy balance (2.21) may be written as

$$-\rho\left(\dot{\epsilon}-\theta\dot{\eta}-\sum_{N=1}^{K}\theta_N\dot{\eta}_N\right)-\rho\left(\theta\xi+\sum_{N=1}^{K}\theta_N\xi_N\right)+\rho\bar{w}-\boldsymbol{p}\cdot\boldsymbol{g}-\sum_{N=1}^{K}\boldsymbol{p}_N\cdot\boldsymbol{g}_N+P=0, \tag{2.25}$$

where the temperature gradients $\boldsymbol{g}$ and $\boldsymbol{g}_N$ are defined by

$$\boldsymbol{g}=\operatorname{grad}_s\theta=\theta_{,\alpha}\boldsymbol{a}^\alpha,\quad \boldsymbol{g}_N=\operatorname{grad}_s\theta_N=\theta_{N,\alpha}\boldsymbol{a}^\alpha \tag{2.26}$$

and
$$P=\boldsymbol{N}\cdot\boldsymbol{L}+\sum_{N=1}^{P}(\boldsymbol{K}^N+\boldsymbol{M}^N)\cdot\boldsymbol{L}_N=\boldsymbol{N}^\alpha\cdot\boldsymbol{v}_{,\alpha}+\sum_{N=1}^{P}(\boldsymbol{k}^N\cdot\boldsymbol{w}_N+\boldsymbol{M}^{N\alpha}\cdot\boldsymbol{w}_{N,\alpha}) \tag{2.27}$$

represents the mechanical power. After suitable choice of constitutive equations (2.25) is to be regarded as an identity for all thermomechanical processes (Green & Naghdi 1977).

Integral balance equations, corresponding to (2.7)–(2.10) and (2.16)–(2.18), in terms of field quantities in a reference state may also be recorded but we do not list these here. We note, however, they may be obtained from the foregoing equations by replacing $\mathscr{P}$, $\partial\mathscr{P}$, $\mathrm{d}\sigma$, $\mathrm{d}s$ with $\mathscr{P}_{\mathrm{R}}$, $\partial\mathscr{P}_{\mathrm{R}}$, $\mathrm{d}\sigma_{\mathrm{R}}$, $\mathrm{d}s_{\mathrm{R}}$, respectively, and by replacing

$$\rho,\ \{\boldsymbol{n},\boldsymbol{m}^N,\boldsymbol{k}^N\},\quad \{h,h_N,k,k_N\}, \tag{2.28}$$

with the corresponding quantities in the reference state, namely

$$\rho_{\mathrm{R}}=\rho(a/A)^{\frac{1}{2}},\quad \{{}_{\mathrm{R}}\boldsymbol{n},{}_{\mathrm{R}}\boldsymbol{m}^N,{}_{\mathrm{R}}\boldsymbol{k}^N\},\quad \{{}_{\mathrm{R}}h,{}_{\mathrm{R}}h_N,{}_{\mathrm{R}}k,{}_{\mathrm{R}}k_N\}, \tag{2.29}$$

respectively. We also note (Naghdi 1972, equations (9.86), (9.87)) that

$$\left.\begin{aligned}\{{}_{\mathrm{R}}\boldsymbol{N}^\alpha,{}_{\mathrm{R}}\boldsymbol{k}^N,{}_{\mathrm{R}}\boldsymbol{M}^{N\alpha}\}&=(\rho_{\mathrm{R}}/\rho)\{\boldsymbol{N}^\alpha,\boldsymbol{k}^N,\boldsymbol{M}^{N\alpha}\},\\ \{{}_{\mathrm{R}}\boldsymbol{N}\boldsymbol{F}^{\mathrm{T}},{}_{\mathrm{R}}\boldsymbol{M}^N\boldsymbol{F}^{\mathrm{T}},\boldsymbol{F}_{\mathrm{R}}\boldsymbol{p},\boldsymbol{F}_{\mathrm{R}}\boldsymbol{p}_N\}&=(\rho_{\mathrm{R}}/\rho)\{\boldsymbol{N},\boldsymbol{M}^N,\boldsymbol{p},\boldsymbol{p}_N\}.\end{aligned}\right\} \tag{2.30}$$

The field equations in the reference state are then given by

$$\rho_{\mathrm{R}}\left(\dot{\boldsymbol{v}}+\sum_{M=1}^{P}y^{M0}\dot{\boldsymbol{w}}_M\right)=\rho_{\mathrm{R}}(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})+\operatorname{Div}_s{}_{\mathrm{R}}\boldsymbol{N}, \tag{2.31}$$

$$\rho_{\mathrm{R}}\left(y^{N0}\dot{\boldsymbol{v}}+\sum_{M=1}^{P}y^{NM}\dot{\boldsymbol{w}}_M\right)=\rho_{\mathrm{R}}(\boldsymbol{l}^N+\boldsymbol{l}^N_{\mathrm{e}})-{}_{\mathrm{R}}\boldsymbol{k}^N+\operatorname{Div}_s{}_{\mathrm{R}}\boldsymbol{M}^N, \tag{2.32}$$

$$\rho_{\mathrm{R}}\boldsymbol{\Gamma}_{\mathrm{e}}+{}_{\mathrm{R}}\boldsymbol{N}\boldsymbol{F}^{\mathrm{T}}-\boldsymbol{F}{}_{\mathrm{R}}\boldsymbol{N}^{\mathrm{T}}+\sum_{N=1}^{P}({}_{\mathrm{R}}\boldsymbol{K}^N\boldsymbol{F}_N^{\mathrm{T}}-\boldsymbol{F}_N{}_{\mathrm{R}}\boldsymbol{K}^{N\mathrm{T}}+{}_{\mathrm{R}}\boldsymbol{M}^N\boldsymbol{G}_N^{\mathrm{T}}-\boldsymbol{G}_N{}_{\mathrm{R}}\boldsymbol{M}^{N\mathrm{T}})=\boldsymbol{0}, \tag{2.33}$$

$$\rho_{\mathrm{R}}\boldsymbol{c}_{\mathrm{e}}+\boldsymbol{a}_\alpha\times{}_{\mathrm{R}}\boldsymbol{N}^\alpha+\sum_{N=1}^{P}(\boldsymbol{d}_N\times{}_{\mathrm{R}}\boldsymbol{k}^N+\boldsymbol{d}_{N,\alpha}\times{}_{\mathrm{R}}\boldsymbol{M}^{N\alpha})=\boldsymbol{0}, \tag{2.33a}$$

$$\rho_{\mathrm{R}}\dot{\eta}=\rho_{\mathrm{R}}(s+\xi)-\operatorname{Div}_s{}_{\mathrm{R}}\boldsymbol{p},\quad \rho_{\mathrm{R}}\dot{\eta}_N=\rho_{\mathrm{R}}(s_N+\xi_N)-\operatorname{Div}_s{}_{\mathrm{R}}\boldsymbol{p}_N, \tag{2.34}$$

$$-\rho_{\mathrm{R}}\left(\dot{\epsilon}-\theta\dot{\eta}-\sum_{N=1}^{K}\theta_N\dot{\eta}_N\right)-\rho_{\mathrm{R}}\left(\theta\xi+\sum_{N=1}^{K}\theta_N\xi_N\right)+\rho_{\mathrm{R}}\bar{w}-{}_{\mathrm{R}}\boldsymbol{p}\cdot{}_{\mathrm{R}}\boldsymbol{g}-\sum_{N=1}^{K}{}_{\mathrm{R}}\boldsymbol{p}\cdot{}_{\mathrm{R}}\boldsymbol{g}_N+P_{\mathrm{R}}=0, \tag{2.35}$$

where
$${}_{\mathrm{R}}\boldsymbol{g}=\operatorname{Grad}_s\theta=\theta_{,\alpha}\boldsymbol{A}^\alpha,\quad {}_{\mathrm{R}}\boldsymbol{g}_N=\operatorname{Grad}_s\theta_N=\theta_{N,\alpha}\boldsymbol{A}^\alpha, \tag{2.36}$$

$$P_{\mathrm{R}}={}_{\mathrm{R}}\boldsymbol{N}\cdot\dot{\boldsymbol{F}}+\sum_{N=1}^{P}({}_{\mathrm{R}}\boldsymbol{K}^N+{}_{\mathrm{R}}\boldsymbol{M}^N)\cdot\dot{\boldsymbol{G}}_N={}_{\mathrm{R}}\boldsymbol{N}^\alpha\cdot\boldsymbol{v}_{,\alpha}+\sum_{N=1}^{P}({}_{\mathrm{R}}\boldsymbol{k}^N\cdot\boldsymbol{w}_N+{}_{\mathrm{R}}\boldsymbol{M}^{N\alpha}\cdot\boldsymbol{w}_{N,\alpha}), \tag{2.37}$$

$${}_{\mathrm{R}}\boldsymbol{n}={}_{\mathrm{R}}\boldsymbol{N}{}_{\mathrm{R}}\boldsymbol{\nu}={}_{\mathrm{R}}\boldsymbol{N}^\alpha{}_{\mathrm{R}}\nu_\alpha,\quad {}_{\mathrm{R}}\boldsymbol{\nu}={}_{\mathrm{R}}\nu_\alpha\boldsymbol{A}^\alpha,\quad {}_{\mathrm{R}}\boldsymbol{N}={}_{\mathrm{R}}\boldsymbol{N}^\alpha\otimes\boldsymbol{A}_\alpha,\quad {}_{\mathrm{R}}\boldsymbol{N}^\alpha={}_{\mathrm{R}}N^{i\alpha}\boldsymbol{A}_i, \tag{2.38}$$

$${}_{\mathrm{R}}\boldsymbol{m}^N={}_{\mathrm{R}}\boldsymbol{M}^N{}_{\mathrm{R}}\boldsymbol{\nu}={}_{\mathrm{R}}\boldsymbol{M}^{N\alpha}{}_{\mathrm{R}}\nu_\alpha,\quad {}_{\mathrm{R}}\boldsymbol{M}^N={}_{\mathrm{R}}\boldsymbol{M}^{N\alpha}\otimes\boldsymbol{A}_\alpha,\quad {}_{\mathrm{R}}\boldsymbol{M}^{N\alpha}={}_{\mathrm{R}}M^{Ni\alpha}\boldsymbol{A}_i, \tag{2.39}$$

$${}_{\mathrm{R}}\boldsymbol{k}^N={}_{\mathrm{R}}k^{Ni}\boldsymbol{A}_i={}_{\mathrm{R}}k_i^N\boldsymbol{A}^i,\quad {}_{\mathrm{R}}\boldsymbol{K}^N={}_{\mathrm{R}}\boldsymbol{k}^N\otimes\boldsymbol{D}_N, \tag{2.40}$$

$${}_{\mathrm{R}}k={}_{\mathrm{R}}\boldsymbol{p}\cdot{}_{\mathrm{R}}\boldsymbol{\nu},\quad {}_{\mathrm{R}}k_N={}_{\mathrm{R}}\boldsymbol{p}_N\cdot{}_{\mathrm{R}}\boldsymbol{\nu} \tag{2.41}$$

and where

$$\left.\begin{aligned} A^{\frac{1}{2}}\,\mathrm{Div}_{s\,\mathrm{R}}\boldsymbol{N} &= (A^{\frac{1}{2}}{}_{\mathrm{R}}\boldsymbol{N}\boldsymbol{A}^{\alpha})_{,\alpha} = (A^{\frac{1}{2}}{}_{\mathrm{R}}\boldsymbol{N}^{\alpha})_{,\alpha}, \\ A^{\frac{1}{2}}\,\mathrm{Div}_{s\,\mathrm{R}}\boldsymbol{M}^{N} &= (A^{\frac{1}{2}}{}_{\mathrm{R}}\boldsymbol{M}^{N}\boldsymbol{A}^{\alpha})_{,\alpha} = (A^{\frac{1}{2}}{}_{\mathrm{R}}\boldsymbol{M}^{N\alpha})_{,\alpha}, \\ A^{\frac{1}{2}}\,\mathrm{Div}_{s\,\mathrm{R}}\boldsymbol{p} = (A^{\frac{1}{2}}{}_{\mathrm{R}}\boldsymbol{p}\cdot\boldsymbol{A}^{\alpha})_{,\alpha}, &\quad A^{\frac{1}{2}}\,\mathrm{Div}_{s\,\mathrm{R}}\boldsymbol{p}_{N} = (A^{\frac{1}{2}}{}_{\mathrm{R}}\boldsymbol{p}_{N}\cdot\boldsymbol{A}^{\alpha})_{,\alpha}. \end{aligned}\right\} \tag{2.42}$$

The various kinematical and kinetical results in this section are recorded for Cosserat surfaces $\mathscr{C}_P$ (with P directors, $P = 1, 2, \ldots, N$). When considering a Cosserat surface with a single director, namely when $\mathscr{C}_1 = \mathscr{C}$, it is convenient to adopt the notations

$$\boldsymbol{d}_1 = \boldsymbol{d}, \quad \boldsymbol{w}_1 = \boldsymbol{w}, \quad \boldsymbol{D}_1 = \boldsymbol{D}, \quad \boldsymbol{d}_{1\alpha} = \boldsymbol{d}_{\alpha}, \quad \boldsymbol{G}_1 = \boldsymbol{G}, \quad \text{etc.,} \tag{2.43}$$

for the kinematical quantities in $(2.1)_2$, $(2.3)_2$, $(2.4)_2$ and (2.6). Correspondingly, for the thermomechanical quantities in (2.7)–(2.10) and (2.13)–(2.18), we write

$$\boldsymbol{m}^1 = \boldsymbol{m}, \quad \boldsymbol{l}^1 = \boldsymbol{l}, \quad \boldsymbol{M}^1 = \boldsymbol{M}, \quad \boldsymbol{M}^{1\alpha} = \boldsymbol{M}^{\alpha}, \quad \boldsymbol{k}^1 = \boldsymbol{k}, \quad \boldsymbol{K}^1 = \boldsymbol{K}. \tag{2.44}$$

3. Electromagnetic equations for Cosserat surfaces

We complete in this section the basic theory of Cosserat surfaces by a direct approach, in the presence of electromagnetic effects. For this purpose, we introduce the appropriate electromagnetic variables guided by the exact three-dimensional developments summarized in Appendixes A and B. We assume that electromagnetic effects are represented by the following field quantities:

$$\left.\begin{array}{l} \text{the electric field vectors, } \boldsymbol{e}^*_N = e^*_{Ni}\,\boldsymbol{a}^i; \\ \text{the electric displacement field vectors†, } \bar{\boldsymbol{d}}_N = \bar{d}^i_N\,\boldsymbol{a}_i; \\ \text{the magnetic field vectors, } \boldsymbol{h}^*_N = h^*_{Ni}\,\boldsymbol{a}^i; \\ \text{the magnetic induction field vectors, } \boldsymbol{b}_N = b^i_N\,\boldsymbol{a}_i; \\ \text{the current density field vectors, } \boldsymbol{j}^*_N = j^{*i}_N\,\boldsymbol{a}_i; \\ \text{the free charge represented by the scalar fields, } e_N; \end{array}\right\} \tag{3.1}$$

all for $N = 0, 1, \ldots, L$.

Having defined the above field quantities, we now proceed to record the appropriate balance equations for a moving shell-like body, which are analogues of the three-dimensional balance equations (A 1)–(A 3) associated with the names of Faraday, Ampere and Gauss. We only record here the main results concerning the electromagnetic equations for Cosserat surfaces and for details refer the reader to Appendix B, particularly equations (B 20)–(B 23). Thus, corresponding to the Gauss equations (A 3) and in view of the developments of Appendix B, we assume that the fields $\boldsymbol{b}_M$, $\bar{\boldsymbol{d}}_M$ satisfy the balance laws

$$\int_{\partial\mathscr{P}} \boldsymbol{b}_M\cdot\boldsymbol{v}\,\mathrm{d}s = \int_{\mathscr{P}} \left(\sum_{K=0}^{M} \chi^K_M\,\boldsymbol{b}_K - \hat{\boldsymbol{b}}_M\right)\cdot\mathrm{d}\boldsymbol{\sigma}, \tag{3.2}$$

$$\int_{\partial\mathscr{P}} \bar{\boldsymbol{d}}_M\cdot\boldsymbol{v}\,\mathrm{d}s = \int_{\mathscr{P}} e_M\,\mathrm{d}\sigma + \int_{\mathscr{P}} \left(\sum_{K=0}^{M} \psi^K_M\,\bar{\boldsymbol{d}}_K - \hat{\boldsymbol{d}}_M\right)\cdot\mathrm{d}\boldsymbol{\sigma}. \tag{3.3}$$

† We use an overbar to designate the electric displacement fields, i.e. $\bar{\boldsymbol{d}}_N$, (rather than the more customary symbol $\boldsymbol{d}_N$) in order to avoid confusion with the notation for the director fields such as $\boldsymbol{d}_N$ in (2.2).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Similarly, we assume the analogues of the Faraday and Ampere laws in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \boldsymbol{b}_M\cdot \mathrm{d}\boldsymbol{\sigma} = -\int_{\partial\mathscr{P}} \boldsymbol{e}_M^*\cdot \mathrm{d}\boldsymbol{r}, \tag{3.4a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} (\boldsymbol{a}_3\times\boldsymbol{b}_M)\times \mathrm{d}\boldsymbol{\sigma} = \int_{\partial\mathscr{P}} (\boldsymbol{e}_M^*\cdot\boldsymbol{a}_3)\,\mathrm{d}\boldsymbol{r} + \int_{\mathscr{P}} \boldsymbol{L}[(\boldsymbol{a}_3\times\boldsymbol{b}_M)\times \mathrm{d}\boldsymbol{\sigma}] + \int_{\mathscr{P}}\left(\hat{\boldsymbol{e}}_M^* - \sum_{K=0}^{M}\chi_M^K \boldsymbol{e}_K^*\right)\times \mathrm{d}\boldsymbol{\sigma}, \tag{3.4b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} \bar{\boldsymbol{d}}_M\cdot \mathrm{d}\boldsymbol{\sigma} = \int_{\partial\mathscr{P}} \boldsymbol{h}_M^*\cdot \mathrm{d}\boldsymbol{r} - \int_{\mathscr{P}} \boldsymbol{j}_M^*\cdot \mathrm{d}\boldsymbol{\sigma}, \tag{3.5a}$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}} (\boldsymbol{a}_3\times\bar{\boldsymbol{d}}_M)\times \mathrm{d}\boldsymbol{\sigma} = \int_{\partial\mathscr{P}} (\boldsymbol{h}_M^*\cdot\boldsymbol{a}_3)\,\mathrm{d}\boldsymbol{r} + \int_{\mathscr{P}}\{(\boldsymbol{a}_3\times\boldsymbol{j}_M^*)\times \mathrm{d}\boldsymbol{\sigma} - \boldsymbol{L}[(\boldsymbol{a}_3\times\bar{\boldsymbol{d}}_M)\times \mathrm{d}\boldsymbol{\sigma}]\} + \int_{\mathscr{P}}\left(\hat{\boldsymbol{h}}_M^* - \sum_{K=0}^{M}\psi_M^K \boldsymbol{h}_K^*\right)\times \mathrm{d}\boldsymbol{\sigma}, \tag{3.5b}$$

where $\mathrm{d}\boldsymbol{\sigma} = \boldsymbol{a}_3\,\mathrm{d}\sigma$ and $\hat{\boldsymbol{b}}_M$, $\hat{\boldsymbol{d}}_M$, $\hat{\boldsymbol{e}}_M^*$, $\hat{\boldsymbol{h}}_M^*$, due to contributions from the major surfaces of the shell, are given by

$$\left.\begin{aligned} \hat{\boldsymbol{b}}_M &= [\chi_M(z)\, b^i \boldsymbol{a}_i g^{\frac{1}{2}}/a^{\frac{1}{2}}]_{z_1}^{z_2}, & \hat{\boldsymbol{d}}_M &= [\psi_M(z)\, \bar{d}^i \boldsymbol{a}_i g^{\frac{1}{2}}/a^{\frac{1}{2}}]_{z_1}^{z_2}, \\ \hat{\boldsymbol{e}}_M^* &= [\chi_M(z)\, e_i^* \boldsymbol{a}^i]_{z_1}^{z_2}, & \hat{\boldsymbol{h}}_M^* &= [\psi_M(z)\, h_i^* \boldsymbol{a}^i]_{z_1}^{z_2}. \end{aligned}\right\} \tag{3.6}$$

Equations (3.2)–(3.5) hold for $M = 0, 1, 2, \ldots, L$. We do not record explicitly equations for conservation of charge, since the equations for conservation of charge in the three-dimensional theory may be derived from (A 1)–(A 3) and thus may be omitted here

The field equations that correspond to (3.2)–(3.5) and may be regarded as Maxwell equations for shell-like bodies are

$$\mathrm{div}_s[(\boldsymbol{a}_3\times\boldsymbol{b}_M)\times\boldsymbol{a}_3] = \mathrm{div}_s\,(b_M^\alpha \boldsymbol{a}_\alpha) = a^{-\frac{1}{2}}(b_M^\alpha a^{\frac{1}{2}})_{,\alpha} = \left(\sum_{K=0}^{M}\chi_M^K \boldsymbol{b}_K - \hat{\boldsymbol{b}}_M\right)\cdot\boldsymbol{a}_3, \tag{3.7}$$

$$\mathrm{div}_s[(\boldsymbol{a}_3\times\bar{\boldsymbol{d}}_M)\times\boldsymbol{a}_3] = \mathrm{div}_s\,(\bar{d}_M^\alpha \boldsymbol{a}_\alpha) = a^{-\frac{1}{2}}(\bar{d}_M^\alpha a^{\frac{1}{2}})_{,\alpha} = \left(\sum_{K=0}^{M}\psi_M^K \bar{\boldsymbol{d}}_K - \hat{\boldsymbol{d}}_M\right)\cdot\boldsymbol{a}_3 + e_M, \tag{3.8}$$

$$\dot{\boldsymbol{b}}_M + \boldsymbol{b}_M\,\mathrm{div}_s\,\boldsymbol{v} - \boldsymbol{L}\boldsymbol{b}_M = -\mathrm{curl}_s\,\boldsymbol{e}_M^* - \boldsymbol{a}_3\times\bar{\boldsymbol{K}}\boldsymbol{e}_M^* - \boldsymbol{a}_3\times\left(\hat{\boldsymbol{e}}_M^* - \sum_{K=0}^{M}\chi_M^K\boldsymbol{e}_K^*\right), \tag{3.9}$$

$$\dot{\bar{\boldsymbol{d}}}_M + \bar{\boldsymbol{d}}_M\,\mathrm{div}_s\,\boldsymbol{v} - \boldsymbol{L}\bar{\boldsymbol{d}}_M = \mathrm{curl}_s\,\boldsymbol{h}_M^* - \boldsymbol{j}_M^* + \boldsymbol{a}_3\times\bar{\boldsymbol{K}}\boldsymbol{h}_M^* + \boldsymbol{a}_3\times\left(\hat{\boldsymbol{h}}_M^* - \sum_{K=0}^{M}\psi_M^K\boldsymbol{h}_K^*\right), \tag{3.10}$$

where $\mathrm{div}_s\,\boldsymbol{v}$ is defined by $(2.18)_1$,

$$\mathrm{curl}_s\,\boldsymbol{e}_M^* = \boldsymbol{a}^\alpha\times\boldsymbol{e}_{M,\alpha}^*, \quad \mathrm{curl}_s\,\boldsymbol{h}_M^* = \boldsymbol{a}^\alpha\times\boldsymbol{h}_{M,\alpha}^* \tag{3.11}$$

and $\bar{\boldsymbol{K}}$ is the surface curvature tensor. The field equation (3.9) corresponds to the two balance equations (3.4 *a*, *b*), and equation (3.10) corresponds to (3.5 *a*, *b*).

Similarly, the material forms of balance equations (3.2)–(3.5) are given by

$$\int_{\partial\mathscr{P}_\mathrm{R}} \boldsymbol{B}_M\cdot{}_\mathrm{R}\boldsymbol{\nu}\,\mathrm{d}s_\mathrm{R} = \int_{\mathscr{P}_\mathrm{R}}\left(\sum_{K=0}^{M}\chi_M^K\boldsymbol{B}_K - \hat{\boldsymbol{B}}_M\right)\cdot \mathrm{d}\boldsymbol{\sigma}_\mathrm{R}, \tag{3.12}$$

$$\int_{\partial\mathscr{P}_\mathrm{R}} \bar{\boldsymbol{D}}_M\cdot{}_\mathrm{R}\boldsymbol{\nu}\,\mathrm{d}s_\mathrm{R} = \int_{\mathscr{P}_\mathrm{R}} E\,\mathrm{d}\sigma_\mathrm{R} + \int_{\mathscr{P}_\mathrm{R}}\left(\sum_{K=0}^{M}\psi_M^K\bar{\boldsymbol{D}}_K - \hat{\boldsymbol{D}}_M\right)\cdot \mathrm{d}\boldsymbol{\sigma}_\mathrm{R}, \tag{3.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_\mathrm{R}} \boldsymbol{B}_M\cdot \mathrm{d}\boldsymbol{\sigma}_\mathrm{R} = -\int_{\partial\mathscr{P}_\mathrm{R}} \boldsymbol{E}_M\cdot \mathrm{d}\boldsymbol{R}, \tag{3.14a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_\mathrm{R}} (\boldsymbol{A}_3\times\boldsymbol{B}_M)\times \mathrm{d}\boldsymbol{\sigma}_\mathrm{R} = \int_{\partial\mathscr{P}_\mathrm{R}} (\boldsymbol{E}_M\cdot\boldsymbol{A}_3)\,\mathrm{d}\boldsymbol{R} + \int_{\mathscr{P}_\mathrm{R}}\left(\hat{\boldsymbol{E}}_M - \sum_{K=0}^{M}\chi_M^K\boldsymbol{E}_K\right)\times \mathrm{d}\boldsymbol{\sigma}_\mathrm{R}, \tag{3.14b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_\mathrm{R}} \bar{\boldsymbol{D}}_M\cdot \mathrm{d}\boldsymbol{\sigma}_\mathrm{R} = \int_{\partial\mathscr{P}_\mathrm{R}} \boldsymbol{H}_M\cdot \mathrm{d}\boldsymbol{R} - \int_{\mathscr{P}_\mathrm{R}} \boldsymbol{J}_M\cdot \mathrm{d}\boldsymbol{\sigma}_\mathrm{R}, \tag{3.15a}$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_{\mathrm{R}}}(\boldsymbol{A}_3\times\bar{\boldsymbol{D}}_M)\times\mathrm{d}\boldsymbol{\sigma}_{\mathrm{R}} = \int_{\partial\mathscr{P}_{\mathrm{R}}}(\boldsymbol{H}_M\cdot\boldsymbol{A}_3)\,\mathrm{d}\boldsymbol{R}+\int_{\mathscr{P}_{\mathrm{R}}}(\boldsymbol{A}_3\times\boldsymbol{J}_M)\times\mathrm{d}\boldsymbol{\sigma}_R$$
$$+\int_{\mathscr{P}_{\mathrm{R}}}\left(\hat{\boldsymbol{H}}_M-\sum_{K=0}^{M}\psi_M^K\boldsymbol{H}_K\right)\times\mathrm{d}\boldsymbol{\sigma}_{\mathrm{R}}, \quad (3.15\,b)$$

where

$$\left.\begin{aligned}\hat{\boldsymbol{B}}_M &= [\chi_M(z)\,B^iA_iG^{\frac{1}{2}}/A^{\frac{1}{2}}]_{z_1}^{z_2}, & \hat{\boldsymbol{D}}_M &= [\psi_M(z)\,\bar{D}^iA_iG^{\frac{1}{2}}/A^{\frac{1}{2}}]_{z_1}^{z_2},\\ \hat{\boldsymbol{E}}_M &= [\chi_M(z)\,E_iA^i]_{z_1}^{z_2}, & \hat{\boldsymbol{H}}_M &= [\psi_M(z)\,H_iA^i]_{z_1}^{z_2}\end{aligned}\right\} \quad (3.16)$$

and the vectors in (3.12)–(3.15) are related to those in (3.1) by equations (B 25) in Appendix B.

The field equations that correspond to (3.2)–(3.5) are

$$\mathrm{Div}_s[(\boldsymbol{A}_3\times\boldsymbol{B}_M)\times\boldsymbol{A}_3] = \mathrm{Div}_s(B_M^\alpha\boldsymbol{A}_\alpha) = A^{-\frac{1}{2}}(B_M^\alpha A^{\frac{1}{2}})_{,\alpha} = \left(\sum_{K=0}^{M}\chi_M^K\boldsymbol{B}_K-\hat{\boldsymbol{B}}_M\right)\cdot\boldsymbol{A}_3, \quad (3.17)$$

$$\mathrm{Div}_s[(\boldsymbol{A}_3\times\bar{\boldsymbol{D}}_M)\times\boldsymbol{A}_3] = \mathrm{Div}_s(\bar{D}_M^\alpha\boldsymbol{A}_\alpha) = A^{-\frac{1}{2}}(\bar{D}_M^\alpha A^{\frac{1}{2}})_{,\alpha} = \left(\sum_{K=0}^{M}\psi_M^K\bar{\boldsymbol{D}}_K-\hat{\boldsymbol{D}}_M\right)\cdot\boldsymbol{A}_3+E_M, \quad (3.18)$$

$$\dot{\boldsymbol{B}}_M = -\mathrm{Curl}_s\boldsymbol{E}_M-\boldsymbol{A}_3\times{}_{\mathrm{R}}\bar{\boldsymbol{K}}\boldsymbol{E}_M-\boldsymbol{A}_3\times\left(\hat{\boldsymbol{E}}_M-\sum_{K=0}^{M}\chi_M^K\boldsymbol{E}_K\right), \quad (3.19)$$

$$\dot{\bar{\boldsymbol{D}}}_M = \mathrm{Curl}_s\boldsymbol{H}_M-\boldsymbol{J}_M+\boldsymbol{A}_3\times{}_{\mathrm{R}}\bar{\boldsymbol{K}}\boldsymbol{H}_M+\boldsymbol{A}_3\times\left(\hat{\boldsymbol{H}}_M-\sum_{K=0}^{M}\psi_M^K\boldsymbol{H}_K\right), \quad (3.20)$$

where

$$\mathrm{Curl}_s\boldsymbol{E}_M = \boldsymbol{A}^\alpha\times\boldsymbol{E}_{M,\alpha}, \quad \mathrm{Curl}_s\boldsymbol{H}_M = \boldsymbol{A}^\alpha\times\boldsymbol{H}_{M,\alpha}. \quad (3.21)$$

For the complete theory summarized in §§2 and 3, we also need explicit expressions for $\bar{w}, \boldsymbol{f}_{\mathrm{e}}, \boldsymbol{c}_{\mathrm{e}}$ (or $\boldsymbol{\Gamma}_{\mathrm{e}}$). In view of (B 4) and (B 26)–(B 31), we adopt the following values for $\bar{w}$, $\boldsymbol{\Gamma}_{\mathrm{e}}$ or $\boldsymbol{c}_{\mathrm{e}}$:

$$\begin{aligned}\rho a^{\frac{1}{2}}\bar{w} &= \rho_{\mathrm{R}}A^{\frac{1}{2}}\bar{w}\\ &= a^{\frac{1}{2}}\left[P_{\mathrm{e}}+\sum_{K=0}^{L}\{\boldsymbol{e}_K^*\cdot\boldsymbol{j}_K^*+\boldsymbol{e}_K^*\cdot(\dot{\bar{\boldsymbol{d}}}_K+\bar{\boldsymbol{d}}_K\,\mathrm{div}_s\boldsymbol{v}-\boldsymbol{L}\bar{\boldsymbol{d}}_K)+\boldsymbol{h}_K^*\cdot(\dot{\boldsymbol{b}}_K+\boldsymbol{b}_K\,\mathrm{div}_s\boldsymbol{v}-\boldsymbol{L}\boldsymbol{b}_K)\}\right]\\ &= A^{\frac{1}{2}}\left\{P_{\mathrm{Re}}+\sum_{K=0}^{L}(\boldsymbol{E}_K\cdot\boldsymbol{J}_K+\boldsymbol{E}_K\cdot\dot{\bar{\boldsymbol{D}}}_K+\boldsymbol{H}_K\cdot\dot{\boldsymbol{B}}_K)\right\},\end{aligned} \quad (3.22)$$

where

$$\left.\begin{aligned}P_{\mathrm{e}} &= \boldsymbol{N}_{\mathrm{e}}\cdot\boldsymbol{L}+\sum_{N=1}^{P}(\boldsymbol{K}_{\mathrm{e}}^N+\boldsymbol{M}_{\mathrm{e}}^N)\cdot\boldsymbol{L}_N = \boldsymbol{N}_{\mathrm{e}}^\alpha\cdot\boldsymbol{v}_{,\alpha}+\sum_{N=1}^{P}(\boldsymbol{k}_{\mathrm{e}}^N\cdot\boldsymbol{w}_N+\boldsymbol{M}_{\mathrm{e}}^{N\alpha}\cdot\boldsymbol{w}_{N,\alpha}),\\ P_{\mathrm{Re}} &= {}_{\mathrm{R}}\boldsymbol{N}_{\mathrm{e}}\cdot\dot{\boldsymbol{F}}+\sum_{N=1}^{P}({}_{\mathrm{R}}\boldsymbol{K}_{\mathrm{e}}^N+{}_{\mathrm{R}}\boldsymbol{M}_{\mathrm{e}}^N)\cdot\dot{\boldsymbol{G}}_N = {}_{\mathrm{R}}\boldsymbol{N}_{\mathrm{e}}^\alpha\cdot\boldsymbol{v}_{,\alpha}+\sum_{N=1}^{P}({}_{\mathrm{R}}\boldsymbol{k}_{\mathrm{e}}^N\cdot\boldsymbol{w}_N+{}_{\mathrm{R}}\boldsymbol{M}_{\mathrm{e}}^{N\alpha}\cdot\boldsymbol{w}_{N,\alpha})\end{aligned}\right\} \quad (3.23)$$

and

$$\begin{aligned}\rho a^{\frac{1}{2}}\boldsymbol{\Gamma}_{\mathrm{e}} &= \rho_{\mathrm{R}}A^{\frac{1}{2}}\boldsymbol{\Gamma}_{\mathrm{e}}\\ &= a^{\frac{1}{2}}\{\boldsymbol{N}_{\mathrm{e}}-\boldsymbol{N}_{\mathrm{e}}^{\mathrm{T}}+\sum_{N=1}^{P}(\boldsymbol{K}_{\mathrm{e}}^N-\boldsymbol{K}_{\mathrm{e}}^{N\mathrm{T}}+\boldsymbol{M}_{\mathrm{e}}^N{}_t\boldsymbol{G}_N^{\mathrm{T}}-{}_t\boldsymbol{G}_N\boldsymbol{M}_{\mathrm{e}}^{N\mathrm{T}})\}\\ &= A^{\frac{1}{2}}\{{}_{\mathrm{R}}\boldsymbol{N}_{\mathrm{e}}\boldsymbol{F}^{\mathrm{T}}-\boldsymbol{F}{}_{\mathrm{R}}\boldsymbol{N}_{\mathrm{e}}^{\mathrm{T}}+\sum_{N=1}^{P}({}_{\mathrm{R}}\boldsymbol{K}_{\mathrm{e}}^N\boldsymbol{F}_N^{\mathrm{T}}-\boldsymbol{F}_N{}_{\mathrm{R}}\boldsymbol{K}_{\mathrm{e}}^{N\mathrm{T}}+{}_{\mathrm{R}}\boldsymbol{M}_{\mathrm{e}}^N\boldsymbol{G}_N^{\mathrm{T}}-\boldsymbol{G}_N{}_{\mathrm{R}}\boldsymbol{M}_{\mathrm{e}}^{N\mathrm{T}})\}\end{aligned} \quad (3.24\,a)$$

or

$$\begin{aligned}\rho a^{\frac{1}{2}}\boldsymbol{c}_{\mathrm{e}} = \rho_{\mathrm{R}}A^{\frac{1}{2}}\boldsymbol{c}_{\mathrm{e}} &= a^{\frac{1}{2}}\{\boldsymbol{a}_\alpha\times\boldsymbol{N}_{\mathrm{e}}^\alpha+\sum_{N=1}^{P}(\boldsymbol{d}_N\times\boldsymbol{k}_{\mathrm{e}}^N+\boldsymbol{d}_{N,\alpha}\times\boldsymbol{M}_{\mathrm{e}}^{N\alpha})\}\\ &= A^{\frac{1}{2}}\{\boldsymbol{a}_\alpha\times{}_{\mathrm{R}}\boldsymbol{N}_{\mathrm{e}}^\alpha+\sum_{N=1}^{P}(\boldsymbol{d}_N\times{}_{\mathrm{R}}\boldsymbol{k}_{\mathrm{e}}^N+\boldsymbol{d}_{N,\alpha}\times{}_{\mathrm{R}}\boldsymbol{M}_{\mathrm{e}}^{N\alpha})\}.\end{aligned} \quad (3.24\,b)$$

We note that under a Galilean transformation of the vector $\boldsymbol{r}$ and directors $\boldsymbol{d}_N$, namely

$$\boldsymbol{r}^+ = \boldsymbol{Q}\boldsymbol{r}+\boldsymbol{r}_0t, \quad \boldsymbol{d}_N^+ = \boldsymbol{Q}\boldsymbol{d}_N, \quad (3.25)$$

where $\boldsymbol{r}_0$ is a constant vector and $\boldsymbol{Q}$ is a constant orthogonal tensor, the following transformations hold for the electromagnetic quantities:

$$\left.\begin{aligned} \bar{\boldsymbol{d}}_M^+ = \boldsymbol{Q}\bar{\boldsymbol{d}}_M, \quad \boldsymbol{b}_M^+ = \boldsymbol{Q}\boldsymbol{b}_M \det \boldsymbol{Q}, \quad \boldsymbol{e}_M^{*+} = \boldsymbol{Q}\boldsymbol{e}_M^*,\\ \boldsymbol{h}_M^{*+} = \boldsymbol{Q}\boldsymbol{h}_M^* \det \boldsymbol{Q}, \quad \boldsymbol{j}_M^{*+} = \boldsymbol{Q}\boldsymbol{j}_M^*, \quad e_M^+ = e_M,\\ \boldsymbol{E}_M^+ = \boldsymbol{E}_M, \quad \boldsymbol{H}_M^+ = \boldsymbol{H}_M \det \boldsymbol{Q}, \quad \boldsymbol{D}_M^+ = \boldsymbol{D}_M \det \boldsymbol{Q}, \quad \boldsymbol{B}^+ = \boldsymbol{B},\\ \boldsymbol{J}_M^+ = \boldsymbol{J}_M \det \boldsymbol{Q}, \quad E_M^+ = E_M \det \boldsymbol{Q}. \end{aligned}\right\} \tag{3.26}$$

When the Cosserat surface is subjected to a constant rigid body velocity and a constant superposed rigid body rotation, the same relations (3.26) hold with $\det \boldsymbol{Q} = 1$.

4. Magnetic, polarized thermoelastic Cosserat surfaces

We begin our discussion of constitutive equations for magnetic, polarized thermoelastic Cosserat surfaces by introducing a specific Helmholtz free energy ψ defined by

$$\begin{aligned} \psi &= \epsilon - \theta\eta - \sum_{N=1}^{K} \theta_N \eta_N - \rho^{-1} \sum_{M=0}^{L} (\boldsymbol{e}_M^* \cdot \bar{\boldsymbol{d}}_M + \boldsymbol{h}_M^* \cdot \boldsymbol{b}_M)\\ &= \epsilon - \theta\eta - \sum_{N=1}^{K} \theta_N \eta_N - \rho_{\mathrm{R}}^{-1} \sum_{M=0}^{L} (\boldsymbol{E}_M \cdot \bar{\boldsymbol{D}}_M + \boldsymbol{H}_M \cdot \boldsymbol{B}_M). \end{aligned} \tag{4.1}$$

Then, with the use of (3.22) and (3.23), the energy equation (2.25) can be reduced to

$$\begin{aligned} -\rho\left(\dot{\psi} + \eta\dot{\theta} + \sum_{N=1}^{K} \eta_N \dot{\theta}_N\right) - \rho\left(\theta\xi + \sum_{N=1}^{K} \theta_N \xi_N\right) - \boldsymbol{p}\cdot\boldsymbol{g} - \sum_{N=1}^{K} \boldsymbol{p}_N \cdot \boldsymbol{g}_N \\ + \sum_{M=0}^{L} (\boldsymbol{e}_M^* \cdot \boldsymbol{j}_M^* - \bar{\boldsymbol{d}}_M \cdot \dot{\boldsymbol{e}}_M^* - \boldsymbol{L}\cdot \boldsymbol{e}_M^* \otimes \bar{\boldsymbol{d}}_M - \boldsymbol{b}_M \cdot \dot{\boldsymbol{h}}_M^* - \boldsymbol{L}\cdot\boldsymbol{h}_M^* \otimes \boldsymbol{b}_M) + P + P_{\mathrm{e}} = 0. \end{aligned} \tag{4.2}$$

The material form of the reduced energy equation can be found by direct transformation of (4.2), or from (2.35), and is given by

$$\begin{aligned} -\rho_{\mathrm{R}}\left(\dot{\psi} + \eta\dot{\theta} + \sum_{N=1}^{K} \eta_N \dot{\theta}_N\right) - \rho_{\mathrm{R}}\left(\theta\xi + \sum_{N=1}^{K} \theta_N \xi_N\right) - {}_{\mathrm{R}}\boldsymbol{p}\cdot{}_{\mathrm{R}}\boldsymbol{g} - \sum_{N=1}^{K} {}_{\mathrm{R}}\boldsymbol{p}_N \cdot {}_{\mathrm{R}}\boldsymbol{g}_N \\ + \sum_{M=0}^{L} (\boldsymbol{E}_M \cdot \boldsymbol{J}_M - \bar{\boldsymbol{D}}_M \cdot \dot{\boldsymbol{E}}_M - \boldsymbol{B}_M \cdot \dot{\boldsymbol{H}}_M) + P_{\mathrm{R}} + P_{\mathrm{Re}} = 0, \end{aligned} \tag{4.3}$$

where P_{R}, P_{Re} are defined by (2.37) and (3.23), respectively.

In a complete theory, in addition to the field equations of § 2 and the electromagnetic equations of § 3, constitutive equations must also be specified for appropriate dynamical, thermal and electromagnetic dependent variables. Here we specify the constitutive equations for a given medium by the set of variables

$$\left.\begin{aligned} \psi, \eta, \eta_N, \xi, \xi_N, \boldsymbol{p}, \boldsymbol{p}_N, \boldsymbol{j}_M^*, \bar{\boldsymbol{d}}_M, \boldsymbol{b}_M,\\ \boldsymbol{N} (\text{or } \boldsymbol{N}^\alpha), \boldsymbol{K}^S (\text{or } \boldsymbol{k}^S), \boldsymbol{M}^S (\text{or } \boldsymbol{M}^{S\alpha}). \end{aligned}\right\} \tag{4.4}$$

Further, in accordance with the recent procedure of Green & Naghdi (1977), we then regard the energy equation (4.2) as an identity to be satisfied for all processes subject to the electromagnetic equations (3.7)–(3.10).

In the presence of electromagnetic and thermal effects, elastic Cosserat surfaces are defined as those whose constitutive response functions depend on the variables

$$\theta, \theta_N, \boldsymbol{g}, \boldsymbol{g}_N, \boldsymbol{F}, \boldsymbol{G}_S, \boldsymbol{e}_M^*, \boldsymbol{h}_M^*; \theta^\alpha, \tag{4.5}$$

for $N = 1, 2, \ldots, K$; $M = 0, 1, 2, \ldots, L$; $S = 1, 2, \ldots, P$; as well as functions of values in the reference configuration, namely

$$\Theta, \quad {}_{\mathrm{R}}\boldsymbol{F}, \quad {}_{\mathrm{R}}\boldsymbol{G}_S. \tag{4.6}$$

In the reference state θ_N is zero and Θ is the constant value of θ. Alternatively, the set of variables (4.5) may be replaced by the equivalent sets

$$\theta,\ \theta_N,\ \boldsymbol{g},\ \boldsymbol{g}_N,\ \boldsymbol{a}_\alpha,\ \boldsymbol{d}_S,\ \boldsymbol{d}_{S,\alpha},\ \boldsymbol{e}_M^*,\ \boldsymbol{h}_M^*, \tag{4.7}$$

or

$$\theta,\ \theta_N,\ {}_{\mathrm{R}}\boldsymbol{g},\ {}_{\mathrm{R}}\boldsymbol{g}_N,\ \boldsymbol{F},\ \boldsymbol{G}_S,\ \boldsymbol{E}_M,\ \boldsymbol{H}_M, \tag{4.8}$$

or

$$\theta,\ \theta_N,\ {}_{\mathrm{R}}\boldsymbol{g},\ {}_{\mathrm{R}}\boldsymbol{g}_N,\ \boldsymbol{a}_\alpha,\ \boldsymbol{d}_S,\ \boldsymbol{d}_{S,\alpha},\ \boldsymbol{E}_M,\ \boldsymbol{H}_M. \tag{4.9}$$

First, we set aside any invariance requirements under superposed rigid body motions, and use the energy equation (4.2) or (4.3) as an identity for all thermomechanical processes subject to the equations (3.7)–(3.10) or (3.17)–(3.20). It follows that ψ must be independent of $\boldsymbol{g}, \boldsymbol{g}_N$ or ${}_{\mathrm{R}}\boldsymbol{g}$, ${}_{\mathrm{R}}\boldsymbol{g}_N$. Then, omitting explicit display of the reference variables (4.6) and θ^α, we obtain

$$\left.\begin{aligned}
&\psi = \psi_1(\theta, \theta_N, \boldsymbol{F}, \boldsymbol{G}_S, \boldsymbol{e}_M^*, \boldsymbol{h}_M^*),\\
&\eta = -\frac{\partial\psi_1}{\partial\theta}, \quad \eta_N = -\frac{\partial\psi_1}{\partial\theta_N}, \quad \bar{\boldsymbol{d}}_M = -\rho\frac{\partial\psi_1}{\partial\boldsymbol{e}_M^*}, \quad \boldsymbol{b}_M = -\rho\frac{\partial\psi_1}{\partial\boldsymbol{h}_M^*},\\
&\boldsymbol{N} + \boldsymbol{N}_{\mathrm{e}} = \rho\frac{\partial\psi_1}{\partial\boldsymbol{F}}\boldsymbol{F}^{\mathrm{T}} + \sum_{M=0}^{L}(\boldsymbol{e}_M^* \otimes \bar{\boldsymbol{d}}_M + \boldsymbol{h}_M^* \otimes \boldsymbol{b}_M),\\
&\boldsymbol{K}^S + \boldsymbol{K}_{\mathrm{e}}^S + \boldsymbol{M}^S + \boldsymbol{M}_{\mathrm{e}}^S = \rho\frac{\partial\psi_1}{\partial\boldsymbol{G}_S}\boldsymbol{F}_S^{\mathrm{T}}.
\end{aligned}\right\} \tag{4.10}$$

In view of the form (2.17) for $\boldsymbol{N}$ we note that ψ_1 is subject to the restriction

$$\rho\frac{\partial\psi_1}{\partial\boldsymbol{F}}\boldsymbol{A}_3 + \sum_{M=0}^{L}(\boldsymbol{e}_M^* \otimes \bar{\boldsymbol{d}}_M + \boldsymbol{h}_M^* \otimes \boldsymbol{b}_M)\,\boldsymbol{a}_3 = \boldsymbol{0}. \tag{4.10a}$$

Alternatively, we have

$$\left.\begin{aligned}
&\psi = \psi_2(\theta, \theta_N, \boldsymbol{a}_\alpha, \boldsymbol{d}_S, \boldsymbol{d}_{S,\alpha}, \boldsymbol{e}_M^*, \boldsymbol{h}_M^*),\\
&\eta = -\frac{\partial\psi_2}{\partial\theta}, \quad \eta_N = -\frac{\partial\psi_2}{\partial\theta_N}, \quad \bar{\boldsymbol{d}}_M = -\rho\frac{\partial\psi_2}{\partial\boldsymbol{e}_M^*}, \quad \boldsymbol{b}_M = -\frac{\partial\psi_2}{\partial\boldsymbol{h}_M^*},\\
&\boldsymbol{N}^\alpha + \boldsymbol{N}_{\mathrm{e}}^\alpha = \rho\frac{\partial\psi_2}{\partial\boldsymbol{a}_\alpha} + \sum_{M=0}^{L}(\boldsymbol{e}_M^* \otimes \bar{\boldsymbol{d}}_M + \boldsymbol{h}_M^* \otimes \boldsymbol{b}_M)\,\boldsymbol{a}^\alpha\\
&\qquad\qquad - \sum_{M=0}^{L}\{(\boldsymbol{e}_M^* \times \bar{\boldsymbol{d}}_M + \boldsymbol{h}_M^* \otimes \boldsymbol{b}_M)\cdot(\boldsymbol{a}^\alpha \otimes \boldsymbol{a}_3)\}\,\boldsymbol{a}_3,\\
&\boldsymbol{k}^S + \boldsymbol{k}_{\mathrm{e}}^S = \rho\frac{\partial\psi_2}{\partial\boldsymbol{d}_S}, \quad \boldsymbol{M}^{S\alpha} + \boldsymbol{M}_{\mathrm{e}}^{S\alpha} = \rho\frac{\partial\psi_2}{\partial\boldsymbol{d}_{S,\alpha}}
\end{aligned}\right\} \tag{4.11}$$

or

$$\left.\begin{aligned}
&\psi = \psi_3(\theta, \theta_N, \boldsymbol{F}, \boldsymbol{G}_S, \boldsymbol{E}_M, \boldsymbol{H}_M),\\
&\eta = -\frac{\partial\psi_3}{\partial\theta}, \quad \eta_N = -\frac{\partial\psi_3}{\partial\theta_N}, \quad \bar{\boldsymbol{D}}_M = -\rho_{\mathrm{R}}\frac{\partial\psi_3}{\partial\boldsymbol{E}_M}, \quad \boldsymbol{B}_M = -\rho_{\mathrm{R}}\frac{\partial\psi_3}{\partial\boldsymbol{H}_M},\\
&{}_{\mathrm{R}}\boldsymbol{N} + {}_{\mathrm{R}}\boldsymbol{N}_{\mathrm{e}} = \rho_{\mathrm{R}}\frac{\partial\psi_3}{\partial\boldsymbol{F}}, \quad {}_{\mathrm{R}}\boldsymbol{M}^S + {}_{\mathrm{R}}\boldsymbol{M}_{\mathrm{e}}^S + {}_{\mathrm{R}}\boldsymbol{K}^S + {}_{\mathrm{R}}\boldsymbol{K}_{\mathrm{e}}^S = \rho_{\mathrm{R}}\frac{\partial\psi_3}{\partial\boldsymbol{G}_S}, \quad \frac{\partial\psi_3}{\partial\boldsymbol{F}}\boldsymbol{A}_3 = \boldsymbol{0},
\end{aligned}\right\} \tag{4.12}$$

or

$$\left.\begin{aligned}
&\psi = \psi_4(\theta, \theta_N, \boldsymbol{a}_\alpha, \boldsymbol{d}_S, \boldsymbol{d}_{S,\alpha}, \boldsymbol{E}_M, \boldsymbol{H}_M),\\
&\eta = -\frac{\partial\psi_4}{\partial\theta}, \quad \eta_N = -\frac{\partial\psi_4}{\partial\theta_N}, \quad \bar{\boldsymbol{D}}_M = -\rho_{\mathrm{R}}\frac{\partial\psi_4}{\partial\boldsymbol{E}_M}, \quad \boldsymbol{B}_M = -\rho_{\mathrm{R}}\frac{\partial\psi_4}{\partial\boldsymbol{H}_M},\\
&{}_{\mathrm{R}}\boldsymbol{N}^\alpha + {}_{\mathrm{R}}\boldsymbol{N}_{\mathrm{e}}^\alpha = \rho_{\mathrm{R}}\frac{\partial\psi_4}{\partial\boldsymbol{a}_\alpha}, \quad {}_{\mathrm{R}}\boldsymbol{k}^S + {}_{\mathrm{R}}\boldsymbol{k}_{\mathrm{e}}^S = \rho_{\mathrm{R}}\frac{\partial\psi_4}{\partial\boldsymbol{d}_S}, \quad {}_{\mathrm{R}}\boldsymbol{M}^{S\alpha} + {}_{\mathrm{R}}\boldsymbol{M}_{\mathrm{e}}^{S\alpha} = \rho_{\mathrm{R}}\frac{\partial\psi_4}{\partial\boldsymbol{d}_{S,\alpha}}.
\end{aligned}\right\} \tag{4.13}$$

Also

$$\left.\begin{aligned} -\rho\left(\theta\xi+\sum_{N=1}^{K}\theta_N\xi_N\right)-\boldsymbol{p}\cdot\boldsymbol{g}-\sum_{N=1}^{K}\boldsymbol{p}_N\cdot\boldsymbol{g}_N+\sum_{M=0}^{L}\boldsymbol{e}_M^*\cdot\boldsymbol{j}_M^* = 0,\\ -\rho_{\mathrm{R}}\left(\theta\xi+\sum_{N=1}^{K}\theta_N\xi_N\right)-{}_{\mathrm{R}}\boldsymbol{p}\cdot{}_{\mathrm{R}}\boldsymbol{g}-\sum_{N=1}^{K}{}_{\mathrm{R}}\boldsymbol{p}_N\cdot{}_{\mathrm{R}}\boldsymbol{g}_N+\sum_{M=0}^{L}\boldsymbol{E}_M\cdot\boldsymbol{J}_M = 0.\end{aligned}\right\} \tag{4.14}$$

The expressions (4.10) and (4.11) must satisfy the moment of momentum equation (2.16) while (4.12) and (4.13) must satisfy (2.33). This will yield restrictions on the form of ψ that are equivalent to requiring that ψ be unaltered under a (static) rigid body rotation. It is, perhaps, simplest to impose this condition on the form ψ_4 in (4.13) since only $\boldsymbol{a}_\alpha$, $\boldsymbol{d}_S$, $\boldsymbol{d}_{S,\alpha}$ are changed by a rigid body rotation and invariance conditions imposed on ψ_4 as a function of $\boldsymbol{a}_\alpha$, $\boldsymbol{d}$, $\boldsymbol{d}_{,\alpha}$ have already been discussed elsewhere (see for example Naghdi 1972; Green & Naghdi 1979) and is easily extended to P directors. It follows that ψ may be reduced to the different form

$$\psi = \psi_5(a_{\alpha\beta}, d_{Si}, d_{Si\alpha}, \theta, \theta_N, \boldsymbol{E}_M, \boldsymbol{H}_M;\ \boldsymbol{A}_\alpha, D_{Si}, D_{Si\alpha}, \Theta;\ \theta^\alpha). \tag{4.15}$$

Then, with the help of (2.2), (2.4), (2.6), (2.17) and (2.30), it follows from (4.13) that

$$\left.\begin{aligned} &k^{Si}+k_{\mathrm{e}}^{Si} = \rho\frac{\partial\psi_5}{\partial d_{Si}}, \quad M^{Si\alpha}+M_{\mathrm{e}}^{Si\alpha} = \rho\frac{\partial\psi_5}{\partial d_{Si\alpha}},\\ &N^{\beta\alpha}+N_{\mathrm{e}}^{\beta\alpha}-\sum_{S=1}^{P}\{(k^{S\alpha}+k_{\mathrm{e}}^{S\alpha})\,d_S{}^{\beta}+(M^{S\alpha\gamma}+M_{\mathrm{e}}^{S\alpha\gamma})\,d_S{}^{\beta}{}_{\gamma}\} = \rho\left(\frac{\partial\psi_5}{\partial a_{\alpha\beta}}+\frac{\partial\psi_5}{\partial a_{\beta\alpha}}\right),\\ &N^{3\alpha}+N_{\mathrm{e}}^{3\alpha}-\sum_{S=1}^{P}\{(k^{S\alpha}+k_{\mathrm{e}}^{S\alpha})\,d_S{}^{3}+(M^{S\alpha\gamma}+M_{\mathrm{e}}^{S\alpha\gamma})\,d_S{}^{3}{}_{\gamma}-(k^{S3}+k_{\mathrm{e}}^{S3})\,d_S{}^{\alpha}\\ &\qquad -(M^{S3\gamma}+M_{\mathrm{e}}^{S3\gamma})\,d_S{}^{\alpha}{}_{\gamma}\} = 0,\\ &\eta = -\frac{\partial\psi_5}{\partial\theta}, \quad \eta_N = -\frac{\partial\psi_5}{\partial\theta_N}, \quad \bar{\boldsymbol{D}}_M = -\rho_{\mathrm{R}}\frac{\partial\psi_5}{\partial\boldsymbol{E}_M}, \quad \boldsymbol{B}_M = -\rho_{\mathrm{R}}\frac{\partial\psi_5}{\partial\boldsymbol{H}_M},\end{aligned}\right\} \tag{4.16}$$

and

$$\left.\begin{aligned} ({}_{\mathrm{R}}k^{Si}+{}_{\mathrm{R}}k_{\mathrm{e}}^{Si})\,A^{\frac{1}{2}} &= (k^{Sk}+k_{\mathrm{e}}^{Sk})\,(\boldsymbol{a}_k\cdot\boldsymbol{A}^i)\,a^{\frac{1}{2}},\\ ({}_{\mathrm{R}}N^{i\alpha}+{}_{\mathrm{R}}N_{\mathrm{e}}^{i\alpha})\,A^{\frac{1}{2}} &= (N^{k\alpha}+N_{\mathrm{e}}^{k\alpha})\,(\boldsymbol{a}_k\cdot\boldsymbol{A}^i)\,a^{\frac{1}{2}},\\ ({}_{\mathrm{R}}M^{Si\alpha}+{}_{\mathrm{R}}M_{\mathrm{e}}^{Si\alpha})\,A^{\frac{1}{2}} &= (M^{Sk\alpha}+M_{\mathrm{e}}^{Sk\alpha})\,(\boldsymbol{a}_k\cdot\boldsymbol{A}^i)\,a^{\frac{1}{2}}.\end{aligned}\right\} \tag{4.17}$$

5. Symmetries

To model the main features of the response of a (three-dimensional) thin shell, restrictions must be imposed on the constitutive equations of the theory of Cosserat surfaces. In the absence of electromagnetic effects, such constitutive restrictions have previously been (Naghdi 1972, Green & Naghdi 1979) discussed in relation to certain geometrical and material symmetries. In particular, it was assumed in the previous discussions that the shell in its reference configuration has the following properties: (i) it is homogeneous and at constant temperature, (ii) it has certain material symmetries and (iii) it is of uniform thickness h, and normals to the middle surface meet the major surfaces of the shell at points equidistant from the middle surface. We adopt a method slightly different from that used in previous work, especially as we need to consider material symmetries that do not necessarily include symmetry with respect to normal directions to the middle surface.

To deal with property (iii), we assume that the Cosserat surfaces $\mathscr{C}_P$ model a (three-dimensional) shell-like body defined by $(\mathrm{B}\,1)_1$ of Appendix B, for the region $-\frac{1}{2}h \leqslant z \leqslant \frac{1}{2}h$. We identify the material surface $\mathscr{S}$ of $\mathscr{C}_P$ with the middle surface $z = 0$ and use the relative kinematic measures $e_{\alpha\beta}$, γ_{Ri}, $\kappa_{Ri\alpha}$ defined by

$$e_{\alpha\beta} = \tfrac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}), \quad \gamma_{Ri} = d_{Ri} - D_{Ri}, \quad \kappa_{Ri\alpha} = d_{Ri\alpha} - D_{Ri\alpha}. \tag{5.1}$$

The specific Helmholtz free energy function (4.15) may then be replaced by

$$\psi = \hat{\psi}(e_{\alpha\beta}, \gamma_{Ri}, \kappa_{Ri\alpha}, \theta, \theta_N, E_{Mi}, H_{Mi}, D_{Ri}, D_{Ri\alpha}, A_{\alpha\beta}), \tag{5.2}$$

and, in view of property (i), we have suppressed θ^μ and the constants h and Θ.

Although, in the present theory of Cosserat surfaces, it may not be possible to relate $\hat{\psi}$ in (5.2) exactly to a corresponding three-dimensional Helmholtz free energy function by a relation of the type (C 10) of Appendix C, we are guided by (C 11) or (C 12) in formulating the appropriate condition to be satisfied by $\hat{\psi}$ in (5.2) in order to model property (iii), a geometrical symmetry. We observe that the properties (C 11), (C 12) depend on the particularly way the functions $\boldsymbol{r}^*$, $\boldsymbol{R}^*$, θ^*, E_i, H_i are represented so, in parallel with Appendix C, we consider two cases (*a*) and (*b*); but, of course, these may not be the only possibilities.

To model the symmetry property (iii) we impose on the function $\hat{\psi}$ in (5.2) the following conditions in line with Appendix C:

case (*a*)

$$\hat{\psi}(e_{\alpha\beta}, (-1)^{R+1}\gamma_{Ri}, (-1)^R\kappa_{Ri\alpha}, \theta, (-1)^S\theta_S, (-1)^M E_{Mi}, (-1)^M H_{Mi}, (-1)^{R+1} D_{Ri}, (-1)^R D_{Ri\alpha}, A_{\alpha\beta})$$
$$= \hat{\psi}(e_{\alpha\beta}, \gamma_{Ri}, \kappa_{Ri\alpha}, \theta, \theta_S, E_{Mi}, H_{Mi}, D_{Ri}, D_{Ri\alpha}, A_{\alpha\beta}); \tag{5.3a}$$

case (*b*)

$$\hat{\psi}(e_{\alpha\beta}, (-1)^{R+1}\gamma_{Ri}, (-1)^R\kappa_{Ri\alpha}, \theta, (-1)^S\theta_S, (-1)^{M+1} E_{M\alpha}, (-1)^M E_{M3}, (-1)^M H_{M\alpha}, (-1)^{M+1} H_{M3}, (-1)^{R+1} D_{Ri}, (-1)^R D_{Ri\alpha}, A_{\alpha\beta})$$
$$= \hat{\psi}(e_{\alpha\beta}, \gamma_{Ri}, \kappa_{Ri\alpha}, \theta, \theta_S, E_{M\alpha}, E_{M3}, H_{M\alpha}, H_{M3}, D_{Ri}, D_{Ri\alpha}, A_{\alpha\beta}). \tag{5.3b}$$

We next consider restrictions to be imposed on $\hat{\psi}$ in (5.2) that arise when a Cosserat surface models a shell-like body, which, in its reference configuration, is a plate of constant thickness h. We choose coordinates θ^α so that $\boldsymbol{A}_i$ become a constant orthonormal set of vectors $\boldsymbol{e}_i$, and we choose the reference configuration to be specified by

$$D_{1\alpha} = 0, \quad D_{13} = 1, \quad D_{Ri} = 0, \quad D_{Ri\alpha} = 0 \quad (R \geqslant 2). \tag{5.4}$$

We also restrict further discussion to the case when only one director is present, so that

$$\boldsymbol{d}_R = \boldsymbol{0} \quad (R \geqslant 2) \tag{5.5}$$

and we replace

$$\gamma_{1i} \quad \text{by} \quad \gamma_i, \quad \kappa_{1i\alpha} \quad \text{by} \quad \kappa_{i\alpha}. \tag{5.6}$$

Previously, when electromagnetic vectors were absent, we considered symmetry conditions to be imposed on ψ when a Cosserat surface models a plate that is either isotropic or orthotropic (Green & Naghdi 1982) with respect to three directions $\boldsymbol{e}_i$. Here we need to consider some symmetry properties, which do not necessarily include material symmetry with respect to directions normal to the middle surface of the plate. However, since electromagnetic vectors also occur in $\hat{\psi}$, we note first the restrictions to be imposed on $\hat{\psi}$ when the plate is orthotropic.

Suppose there is material symmetry with respect to the normal direction $\boldsymbol{A}_3 = \boldsymbol{e}_3$ to the middle surface of the plate. Then, guided by (C 13) and (C 14) in Appendix C, we assume that $\hat{\psi}$ satisfies the conditions

case (*a*)

$$\hat{\psi}(e_{\alpha\beta}, -\gamma_\alpha, \gamma_3, \kappa_{\alpha\beta}, -\kappa_{3\alpha}, \theta, \theta_S, E_{M\alpha}, -E_{M3}, -H_{M\alpha}, H_{M3}) = \hat{\psi}(e_{\alpha\beta}, \gamma_\alpha, \gamma_3, \kappa_{\alpha\beta}, \kappa_{3\alpha}, \theta, \theta_S, E_{M\alpha}, E_{M3}, H_{M\alpha}, H_{M3}); \quad (5.7a)$$

case (*b*)

$$\hat{\psi}(e_{\alpha\beta}, -\gamma_\alpha, \gamma_3, \kappa_{\alpha\beta}, -\kappa_{3\alpha}, \theta, \theta_S, E_{M\alpha}, -E_{M3}, -H_{M\alpha}, H_{M3}) = \hat{\psi}(e_{\alpha\beta}, \gamma_\alpha, \gamma_3, \kappa_{\alpha\beta}, \kappa_{3\alpha}, \theta, \theta_S, E_{M\alpha}, E_{M3}, H_{M\alpha}, H_{M3}). \quad (5.7b)$$

As far as material symmetries in directions in the plane of the plate are concerned, the conditions imposed on $\hat{\psi}$ are independent of the particular representations used for the Cosserat surface $\mathscr{C}$. If there is material symmetry with respect to the direction $\boldsymbol{e}_1$ in the plate, then we assume that $\hat{\psi}$ satisfies the condition

$$\hat{\psi}(e_{11}, -e_{12}, e_{22}, \gamma_3, -\gamma_1, \gamma_2, -\kappa_{31}, \kappa_{32}, \kappa_{11}, -\kappa_{12}, -\kappa_{21}, \kappa_{22}, \theta, \theta_N, -E_{M1}, E_{M2}, E_{M3}, H_{M1}, -H_{M2}, -H_{M3})$$
$$= \hat{\psi}(e_{11}, e_{12}, e_{22}, \gamma_3, \gamma_1, \gamma_2, \kappa_{31}, \kappa_{32}, \kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22}, \theta, \theta_N, E_{M1}, E_{M2}, E_{M3}, H_{M1}, H_{M2}, H_{M3}). \quad (5.8)$$

If there is material symmetry with respect to the direction $\boldsymbol{e}_2$ in the plate, then $\hat{\psi}$ is assumed to satisfy the condition

$$\hat{\psi}(e_{11}, -e_{12}, e_{22}, \gamma_3, \gamma_1, -\gamma_2, \kappa_{31}, -\kappa_{32}, \kappa_{11}, -\kappa_{12}, -\kappa_{21}, \kappa_{22}, \theta, \theta_N, E_{M1}, -E_{M2}, E_{M3}, -H_{M1}, H_{M2}, -H_{M3})$$
$$= \hat{\psi}(e_{11}, e_{12}, e_{22}, \gamma_3, \gamma_1, \gamma_2, \kappa_{31}, \kappa_{32}, \kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22}, \theta, \theta_N, E_{M1}, E_{M2}, E_{M3}, H_{M1}, H_{M2}, H_{M3}). \quad (5.9)$$

Hence, if the plate is orthotropic with respect to the orthogonal directions $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, then conditions (5.7*a*) or (5.7*b*), (5.8), (5.9) must hold, in addition to (5.3*a*) or (5.3*b*). In the particular case when M has the values 0 and 1 and $N = 1$, and if $\hat{\psi}$ is restricted to be a quadratic form, $\hat{\psi}$ must be a linear combination of the following functions:

case (*a*)

$$\begin{gathered} e_{11}^2, e_{11}e_{22}, e_{22}^2, e_{12}^2, e_{11}\gamma_3, e_{22}\gamma_3, \gamma_3^2, \gamma_1^2, \gamma_2^2, \\ \kappa_{31}^2, \kappa_{32}^2, \kappa_{11}^2, \kappa_{11}\kappa_{22}, \kappa_{12}^2, \kappa_{12}\kappa_{21}, \kappa_{21}^2, \kappa_{22}^2, \\ e_{11}\theta, e_{22}\theta, \gamma_3\theta, \kappa_{11}\theta_1, \kappa_{22}\theta_1, \theta_1^2, \theta^2, \\ E_{01}^2, E_{02}^2, E_{03}^2, E_{11}^2, E_{12}^2, E_{13}^2, \\ H_{01}^2, H_{02}^2, H_{03}^2, H_{11}^2, H_{12}^2, H_{13}^2, \\ e_{12}H_{03}, \gamma_1 H_{02}, \gamma_2 H_{01}, \kappa_{32}H_{11}, \kappa_{31}H_{12}, \kappa_{12}H_{13}, \kappa_{21}H_{13}. \end{gathered} \quad (5.10)$$

In the absence of the electromagnetic vectors, this agrees with the quadratic form used previously for orthotropic plates.

One of the important applications of plate theory is for a plate made of rotated Y-cut quartz (see Tiersten 1969, pp. 53, 162). Such a plate has trigonal-trapezohedral symmetry with x_1 the trigonal axis and x_3 a diagonal axis. A part of the condition of trigonal-trapezohedral symmetry is that there is symmetry with respect to rotation about the $\boldsymbol{e}_2$ direction through an angle π. This leads in case (*a*) to the following conditions:

$$\begin{aligned}&\psi(e_{11}, -e_{12}, e_{22}, \gamma_3, \gamma_1, -\gamma_2, \kappa_{31}, -\kappa_{32}, \kappa_{11}, -\kappa_{12}, -\kappa_{21}, \kappa_{22}, \theta, \theta_N, -E_{M1}, E_{M2}, -E_{M3},\\ &\qquad\qquad -H_{M1}, H_{M2}, -H_{M3})\\ &= \psi(e_{11}, e_{12}, e_{22}, \gamma_3, \gamma_1, \gamma_2, \kappa_{31}, \kappa_{32}, \kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22}, \theta, \theta_N, E_{M1}, E_{M2}, E_{M3}, H_{M1}, H_{M2}, H_{M3}).\end{aligned} \tag{5.11}$$

When $M = 0, 1$ and $N = 1$, and $\hat{\psi}$ is restricted to be a quadratic form, under the conditions (5.3*a*) and (5.11), $\hat{\psi}$ must be a linear combination of the following:

$$\begin{aligned}&e_{11}^2, e_{22}^2, e_{12}^2, e_{11}e_{22}, e_{11}\gamma_3, e_{22}\gamma_3, \gamma_3^2, e_{11}\gamma_1, e_{22}\gamma_1, e_{12}\gamma_2,\\ &\gamma_1\gamma_3, \gamma_1^2, \gamma_2^2, \kappa_{31}^2, \kappa_{31}\kappa_{11}, \kappa_{31}\kappa_{22}, \kappa_{32}^2, \kappa_{32}\kappa_{12}, \kappa_{32}\kappa_{21},\\ &\kappa_{11}^2, \kappa_{11}\kappa_{22}, \kappa_{12}^2, \kappa_{12}\kappa_{21}, \kappa_{21}^2, \kappa_{22}^2,\\ &e_{11}\theta, e_{22}\theta, \gamma_3\theta, \gamma_1\theta, \theta^2, \kappa_{31}\theta_1, \kappa_{11}\theta_1, \kappa_{22}\theta_1, \theta_1^2,\\ &e_{11}E_{02}, e_{22}E_{02}, \gamma_3E_{02}, \gamma_1E_{02}, e_{12}E_{01}, e_{12}E_{03},\\ &\gamma_2E_{01}, \gamma_2E_{03}, E_{01}^2, E_{02}^2, E_{03}^2, E_{01}E_{03},\\ &\kappa_{32}E_{11}, \kappa_{12}E_{11}, \kappa_{21}E_{11}, \kappa_{31}E_{12}, \kappa_{11}E_{12}, \kappa_{22}E_{12},\\ &\kappa_{32}E_{13}, \kappa_{12}E_{13}, \kappa_{21}E_{13},\\ &\theta E_{02}, \theta_1E_{12},\\ &e_{11}H_{02}, e_{22}H_{02}, \gamma_3H_{02}, \gamma_1H_{02}, e_{12}H_{01}, e_{12}H_{03},\\ &\gamma_2H_{01}, \gamma_2H_{03}, H_{01}^2, H_{02}^2, H_{03}^2, H_{01}H_{03},\\ &E_{01}^2, E_{01}E_{03}, E_{01}H_{01}, E_{01}H_{03}, E_{02}^2, E_{02}H_{02}, E_{03}^2, E_{03}H_{01}, E_{03}H_{03}, H_{01}^2, H_{01}H_{03}, H_{02}^2, H_{03}^2,\\ &E_{11}^2, E_{11}E_{13}, E_{11}H_{11}, E_{11}H_{13}, E_{12}^2, E_{12}H_{12}, E_{13}^2, E_{13}H_{11}, E_{13}H_{13}, H_{11}^2, H_{11}H_{13}, H_{12}^2, H_{13}^2,\\ &\kappa_{32}H_{11}, \kappa_{12}H_{11}, \kappa_{21}H_{11}, \kappa_{31}H_{12}, \kappa_{11}H_{12}, \kappa_{22}H_{12},\\ &\kappa_{32}H_{13}, \kappa_{12}H_{13}, \kappa_{21}H_{13},\\ &\theta H_{02}, \theta_1H_{12}.\end{aligned} \tag{5.12}$$

6. Linear theory of a Cosserat plate

We consider in this section the linear theory of a magnetic, polarized, thermoelastic Cosserat surface $\mathscr{C}$ for a plate of a uniform thickness h. In its unstressed reference configuration, the plate is at a uniform temperature $\bar{\theta}$ and there is no electromagnetic field present. The linearization procedure and resulting equations are already available for a thermoelastic Cosserat plate. Extension of this to include linearization of the electro-magnetic aspects of the theory follows similar lines, so we omit details and only record final results.

In the thermomechanical part of the theory, we consider a Cosserat surface with a single director $\boldsymbol{d}_1$ and two temperatures θ and θ_1. For convenience, we adopt the notation used previously (Green & Naghdi 1979), replace θ by $\bar{\theta}+\theta$, and with (2.43) write

$$\boldsymbol{d}_1 = \boldsymbol{d}, \quad \theta_1 = \phi. \tag{6.1}$$

Throughout this section, we use Cartesian tensor notation with all indices written as suffices, and we specify the material surface and the director of $\mathscr{C}$ in the reference configuration by

$$\boldsymbol{R} = x_\alpha \boldsymbol{e}_\alpha, \quad \boldsymbol{D}_1 = \boldsymbol{D} = \boldsymbol{e}_3, \quad A_{\alpha\beta} = \delta_{\alpha\beta}, \quad A = 1. \tag{6.2}$$

The coordinates θ^α are now identified with the rectangular Cartesian coordinates x_α, and $\boldsymbol{e}_i$ is a constant orthonormal system of vectors. Partial differentiation with respect to x_α is denoted by

$(\;)_{,\alpha}$ and the Cartesian summation convention is adopted for repeated suffices. All quantities in various equations and results now refer to the reference configuration (6.2). The material surface of $\mathscr{C}$ is identified with the plane mid-way between the major surfaces $\theta^3 = z = \pm\frac{1}{2}h$ of the plate.

The motion of the Cosserat plate as given by (2.2) and (2.3) is now replaced by

$$\boldsymbol{r} = \boldsymbol{R}+\boldsymbol{u}, \quad \boldsymbol{d} = \boldsymbol{D}+\boldsymbol{\delta}, \quad \boldsymbol{u} = u_i\boldsymbol{e}_i, \quad \boldsymbol{\delta} = \delta_i\boldsymbol{e}_i, \quad \boldsymbol{v} = \dot{\boldsymbol{u}}, \quad \boldsymbol{w} = \dot{\boldsymbol{\delta}}, \tag{6.3}$$

and linear kinematical measures calculated from (6.3) are

$$e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha}), \quad \gamma_3 = \delta_3, \quad \gamma_\alpha = \delta_\alpha+u_{3,\alpha}, \quad \kappa_{\alpha\beta} = \delta_{\alpha,\beta}, \quad \kappa_{3\alpha} = \delta_{3,\alpha}. \tag{6.4}$$

Within the order of approximation of the linearized theory, from §2 and equation (6.3) we have

$$\boldsymbol{\nu} = {}_{\mathrm{R}}\boldsymbol{\nu}, \quad \boldsymbol{n} = {}_{\mathrm{R}}\boldsymbol{n}, \quad \boldsymbol{m} = {}_{\mathrm{R}}\boldsymbol{m}, \quad \boldsymbol{N} = {}_{\mathrm{R}}\boldsymbol{N}, \quad \boldsymbol{M} = {}_{\mathrm{R}}\boldsymbol{M}, \quad \boldsymbol{k} = {}_{\mathrm{R}}\boldsymbol{k},$$

with similar formulae relating $\boldsymbol{N}^\alpha$ to ${}_{\mathrm{R}}\boldsymbol{N}^\alpha$, etc. We therefore omit the suffixes R but understand that all response functions are defined with respect to the reference configuration. Then, from (2.31)–(2.33), field equations of motion are

$$\left.\begin{aligned} \rho(\ddot{\boldsymbol{u}}+y^{10}\ddot{\boldsymbol{\delta}}) &= \rho\boldsymbol{f}+\mathrm{Div}_s\,\boldsymbol{N},\\ (y^{10}\ddot{\boldsymbol{u}}+y^{11}\ddot{\boldsymbol{\delta}}) &= \rho\boldsymbol{l}-\boldsymbol{k}+\mathrm{Div}_s\,\boldsymbol{M},\\ \boldsymbol{N}-\boldsymbol{N}^{\mathrm{T}}+\boldsymbol{K}-\boldsymbol{K}^{\mathrm{T}} = \boldsymbol{0} \quad &\text{or} \quad \boldsymbol{e}_\alpha\times\boldsymbol{N}_\alpha+\boldsymbol{e}_3\times\boldsymbol{k} = \boldsymbol{0}, \end{aligned}\right\} \tag{6.5}$$

where second order terms due to the electromagnetic fields are omitted and where as in (2.44) the fields $\boldsymbol{k}$, $\boldsymbol{M}$, $\boldsymbol{K}$ are written for $\boldsymbol{k}^1$, $\boldsymbol{M}^1$, $\boldsymbol{K}^1$, respectively. Also

$$\left.\begin{aligned} \boldsymbol{n} = \boldsymbol{N}\boldsymbol{\nu} = \boldsymbol{N}_\alpha\nu_\alpha, \quad \boldsymbol{N} = \boldsymbol{N}_\alpha\otimes\boldsymbol{e}_\alpha, \quad \boldsymbol{N}_\alpha = N_{i\alpha}\boldsymbol{e}_i,\\ \boldsymbol{m} = \boldsymbol{M}\boldsymbol{\nu} = \boldsymbol{M}_\alpha\nu_\alpha, \quad \boldsymbol{M} = \boldsymbol{M}_\alpha\otimes\boldsymbol{e}_\alpha, \quad \boldsymbol{M}_\alpha = M_{i\alpha}\boldsymbol{e}_i,\\ \boldsymbol{k} = k_i\boldsymbol{e}_i, \quad \boldsymbol{\nu} = \nu_\alpha\boldsymbol{e}_\alpha, \quad \boldsymbol{f} = f_i\boldsymbol{e}_i, \quad \boldsymbol{l} = l_i\boldsymbol{e}_i,\\ \mathrm{Div}_s\,\boldsymbol{N} = (\boldsymbol{N}\boldsymbol{e}_\alpha)_{,\alpha} = \boldsymbol{N}_{\alpha,\alpha} = N_{i\alpha,\alpha}\boldsymbol{e}_i,\\ \mathrm{Div}_s\,\boldsymbol{M} = (\boldsymbol{M}\boldsymbol{e}_\alpha)_{,\alpha} = \boldsymbol{M}_{\alpha,\alpha} = M_{i\alpha,\alpha}\boldsymbol{e}_i. \end{aligned}\right\} \tag{6.6}$$

Because of the particular choice of the material surface of $\mathscr{C}$ in relation to the major surfaces of the plate we have $y^{10} = 0$, and the component forms of (6.5) reduce to

$$\rho\ddot{u}_i = \rho f_i+N_{i\alpha,\alpha}, \quad \rho y^{11}\ddot{\delta}_i = \rho l_i-k_i+M_{i\alpha,\alpha}, \quad N_{\alpha\beta} = N_{\beta\alpha}, \quad k_\alpha = N_{3\alpha}. \tag{6.7}$$

The field equations of entropy balance that correspond to temperatures θ, ϕ are

$$\left.\begin{aligned} \rho\dot{\eta} &= \rho(s+\xi)-\mathrm{Div}_s\boldsymbol{p}, \quad \mathrm{Div}_s\boldsymbol{p} = p_{\alpha,\alpha},\\ \rho\dot{\eta}_1 &= \rho(s_1+\xi_1)-\mathrm{Div}_s\boldsymbol{p}_1, \quad \mathrm{Div}_s\boldsymbol{p}_1 = p_{1\alpha,\alpha}, \end{aligned}\right\} \tag{6.8}$$

where

$$\left.\begin{aligned} k = \boldsymbol{p}\cdot\boldsymbol{\nu}, \quad k_1 = \boldsymbol{p}_1\cdot\boldsymbol{\nu}, \quad \boldsymbol{p} = p_\alpha\boldsymbol{e}_\alpha, \quad \boldsymbol{p}_1 = p_{1\alpha}\boldsymbol{e}_\alpha,\\ \boldsymbol{q} = \overline{\theta}\boldsymbol{p}, \quad q_1 = 0, \quad r = \overline{\theta}s, \quad r_1 = 0. \end{aligned}\right\} \tag{6.9}$$

Turning to the electromagnetic fields, in view of linearization we have

$$\boldsymbol{E}_N = \boldsymbol{e}^*_N, \quad \boldsymbol{H}_N = \boldsymbol{h}^*_N, \quad \boldsymbol{B}_N = \boldsymbol{b}_N, \quad \overline{\boldsymbol{D}}_N = \overline{\boldsymbol{d}}_N, \quad \boldsymbol{J}_N = \boldsymbol{j}^*_N, \quad E_N = e_N \tag{6.10}$$

and, from (3.17)–(3.20), we see that the field equations are†

$$\left.\begin{aligned} B_{M\alpha,\alpha} = \sum_{K=0}^{M}\chi^K_M B_{K3}-\hat{B}_{M3}, \quad \overline{D}_{M\alpha,\alpha} = \sum_{K=0}^{M}\psi^K_M\overline{D}_{K3}-\hat{D}_{M3}+E^*_M,\\ \dot{\boldsymbol{B}}_M = -\boldsymbol{e}_\alpha\times\boldsymbol{e}_i E_{Mi,\alpha}+\left(\hat{\boldsymbol{E}}_M-\sum_{K=0}^{M}\chi^K_M\boldsymbol{E}_K\right)\times\boldsymbol{e}_3,\\ -\dot{\overline{\boldsymbol{D}}}_M = \boldsymbol{J}_M-\boldsymbol{e}_\alpha\times\boldsymbol{e}_i H_{Mi,\alpha}+\left(\hat{\boldsymbol{H}}_M-\sum_{K=0}^{M}\psi^K_M\boldsymbol{H}_K\right)\times\boldsymbol{e}_3, \end{aligned}\right\} \tag{6.11}$$

† We have written E^*_M for charges E_M in (6.11) to avoid later confusion with components E_i of electric field.

where
$$\left.\begin{aligned} &\boldsymbol{E}_M = E_{Mi}\boldsymbol{e}_i, \quad \boldsymbol{H}_M = H_{Mi}\boldsymbol{e}_i, \quad \bar{\boldsymbol{D}}_M = \bar{D}_{Mi}\boldsymbol{e}_i, \quad \boldsymbol{B}_M = B_{Mi}\boldsymbol{e}_i,\\ &\hat{B}_{M3} = [\chi_M(z)\,B_3]_{z_1}^{z_2}, \quad \hat{D}_{M3} = [\psi_M(z)\,\bar{D}_3]_{z_1}^{z_2}, \quad \boldsymbol{J}_M = J_{Mi}\boldsymbol{e}_i,\\ &\hat{\boldsymbol{E}}_M = [\chi_M(z)\,E_i\boldsymbol{e}_i]_{z_1}^{z_2}, \quad \hat{\boldsymbol{H}}_M = [\psi_M(z)\,H_i\boldsymbol{e}_i]_{z_1}^{z_2}. \end{aligned}\right\} \tag{6.12}$$

Finally, from (4.4), the energy balance equation reduces to
$$-\rho(\dot{\psi}+\eta\dot{\theta}+\eta_1\dot{\phi})-\rho\{(\bar{\theta}+\theta)\,\xi+\phi\xi_1\}-\boldsymbol{p}\cdot\boldsymbol{g}-\boldsymbol{p}_1\cdot\boldsymbol{g}_1+\sum_{M=0}^{L}(\boldsymbol{E}_M\cdot\boldsymbol{J}_M-\bar{\boldsymbol{D}}_M\cdot\dot{\boldsymbol{E}}_M-\boldsymbol{B}_M\cdot\dot{\boldsymbol{H}}_M)$$
$$+\boldsymbol{N}_\alpha\cdot\dot{\boldsymbol{u}}_{,\alpha}+\boldsymbol{k}\cdot\dot{\boldsymbol{\delta}}+\boldsymbol{M}_\alpha\cdot\dot{\boldsymbol{\delta}}_{,\alpha}=0, \tag{6.13}$$
where
$$\boldsymbol{g} = \theta_{,\alpha}\boldsymbol{e}_\alpha, \quad \boldsymbol{g}_1 = \phi_{,\alpha}\boldsymbol{e}_\alpha. \tag{6.14}$$

Then, for a magnetic polarized thermoelastic plate in the linearized theory, we have, either from (6.13) and $(6.7)_{3,4}$ or from the results in (4.16),
$$\left.\begin{aligned} &\psi = \bar{\psi}(e_{\alpha\beta},\gamma_3,\gamma_\alpha,\kappa_{\alpha\beta},\kappa_{3\alpha},\theta,\phi,E_{Mi},H_{Mi}),\\ &k_3 = \rho\frac{\partial\bar{\psi}}{\partial\gamma_3}, \quad k_\alpha = \rho\frac{\partial\bar{\psi}}{\partial\gamma_\alpha}, \quad M_{3\alpha} = \rho\frac{\partial\bar{\psi}}{\partial\kappa_{3\alpha}},\\ &M_{\alpha\beta} = \rho\frac{\partial\bar{\psi}}{\partial\kappa_{\alpha\beta}}, \quad N_{\alpha\beta} = N_{\beta\alpha} = \tfrac{1}{2}\rho\left(\frac{\partial\bar{\psi}}{\partial e_{\alpha\beta}}+\frac{\partial\bar{\psi}}{\partial e_{\beta\alpha}}\right),\\ &\eta = -\frac{\partial\bar{\psi}}{\partial\theta}, \quad \eta_1 = -\frac{\partial\bar{\psi}}{\partial\phi}, \quad \bar{D}_{Mi} = -\rho\frac{\partial\bar{\psi}}{\partial E_{Mi}}, \quad B_{Mi} = -\rho\frac{\partial\bar{\psi}}{\partial H_{Mi}},\\ &-\rho\{(\bar{\theta}+\theta)\,\xi+\phi\xi_1\}-\boldsymbol{p}\cdot\boldsymbol{g}-\boldsymbol{p}_1\cdot\boldsymbol{g}_1+\sum_{M=0}^{L}\boldsymbol{E}_M\cdot\boldsymbol{J}_M = 0. \end{aligned}\right\} \tag{6.15}$$

In view of the conditions (i) and (iii) in § 5, the energy function $\bar{\psi}$ does not depend explicitly on x_α and it satisfies either the invariance condition (5.3*a*) or the condition (5.3*b*). Since the plate is at constant temperature, unstressed and without electromagnetic fields in its reference configuration, $\bar{\psi}$ is a quadratic function of the variables in $(6.15)_1$. In writing down the quadratic for $\bar{\psi}$ we only satisfy the common features of the conditions (5.3*a*, *b*). In each case *a*, *b*, when treated separately, some of the coefficients will be zero. Thus
$$\begin{aligned} \rho\psi = {}&\tfrac{1}{2}A_{\alpha\beta\lambda\mu}e_{\alpha\beta}e_{\lambda\mu}+A_{\alpha\beta\lambda}e_{\alpha\beta}\gamma_\lambda+\tfrac{1}{2}\bar{A}_{\alpha\beta}\gamma_\alpha\gamma_\beta+\tfrac{1}{2}\bar{A}\gamma_3^2+A_\alpha\gamma_\alpha\gamma_3+A'_{\alpha\beta}e_{\alpha\beta}\gamma_3\\ &+\tfrac{1}{2}B_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu}+B_{\alpha\beta\lambda}\kappa_{\alpha\beta}\kappa_{3\lambda}+\tfrac{1}{2}B_{\alpha\beta}\kappa_{3\alpha}\kappa_{3\beta}-\tfrac{1}{2}P\theta^2-P_{\alpha\beta}e_{\alpha\beta}\theta-P_\alpha\gamma_\alpha\theta-R\gamma_3\theta\\ &-\tfrac{1}{2}Q\phi^2-Q_{\alpha\beta}\kappa_{\alpha\beta}\phi-Q_\alpha\kappa_{3\alpha}\phi-e_{\alpha\beta}\sum_{M=0}^{L}(C_{\alpha\beta i}^M E_{Mi}+F_{\alpha\beta i}^M H_{Mi})\\ &-\gamma_\alpha\sum_{M=0}^{L}(C_{\alpha i}^M E_{Mi}+F_{\alpha i}^M H_{Mi})-\gamma_3\sum_{M=0}^{L}(C_i^M E_{Mi}+F_i^M H_{Mi})\\ &-\kappa_{\alpha\beta}\sum_{M=0}^{L}(\bar{C}_{\alpha\beta i}^M E_{Mi}+\bar{F}_{\alpha\beta i}^M H_{Mi})-\kappa_{3\alpha}\sum_{M=0}^{L}(\bar{C}_{\alpha i}^M E_{Mi}+\bar{F}_{\alpha i}^M H_{Mi})\\ &+\theta\sum_{M=0}^{L}(R_i^M E_{Mi}+S_i^M H_{Mi})+\phi\sum_{M=0}^{L}(\bar{R}_i^M E_{Mi}+\bar{S}_i^M H_{Mi})\\ &-\sum_{M=0}^{L}\sum_{N=0}^{L}\{\tfrac{1}{2}L_{ij}^{MN}E_{Mi}E_{Nj}+M_{ij}^{MN}E_{Mi}H_{Nj}+\tfrac{1}{2}N_{ij}^{MN}H_{Mi}H_{Nj}\}. \end{aligned} \tag{6.16}$$

All the coefficients in (6.16) are constants and they have the following properties:
$$\left.\begin{aligned} &A_{\alpha\beta\lambda\mu} = A_{\beta\alpha\lambda\mu} = A_{\alpha\beta\mu\lambda} = A_{\lambda\mu\alpha\beta}, \quad A_{\alpha\beta\lambda} = A_{\beta\alpha\lambda},\\ &\bar{A}_{\alpha\beta} = \bar{A}_{\beta\alpha}, \quad A'_{\alpha\beta} = A'_{\beta\alpha}, \quad B_{\alpha\beta\lambda\mu} = B_{\lambda\mu\alpha\beta}, \quad B_{\alpha\beta} = B_{\beta\alpha},\\ &C_{\alpha\beta i}^M = C_{\beta\alpha i}^M, \quad F_{\alpha\beta i}^M = F_{\beta\alpha i}^M, \quad P_{\alpha\beta} = P_{\beta\alpha}, \quad L_{ij}^{MN} = L_{ji}^{NM}, \quad N_{ij}^{MN} = N_{ji}^{NM}. \end{aligned}\right\} \tag{6.17}$$

From (6.15) and (6.16) it follows that

$$\left.\begin{aligned}
N_{\alpha\beta} = N_{\beta\alpha} &= A_{\alpha\beta\lambda\mu}e_{\lambda\mu} + A_{\alpha\beta\lambda}\gamma_\lambda + A'_{\alpha\beta}\gamma_3 - P_{\alpha\beta}\theta - \sum_{M=0}^{L}(C^M_{\alpha\beta i}E_{Mi} + F^M_{\alpha\beta i}H_{Mi}),\\
k_\alpha &= A_{\lambda\mu\alpha}e_{\lambda\mu} + \bar{A}_{\alpha\beta}\gamma_\beta + A_\alpha\gamma_3 - P_\alpha\theta - \sum_{M=0}^{L}(C^M_{\alpha i}E_{Mi} + F^M_{\alpha i}H_{Mi}),\\
k_3 &= A'_{\alpha\beta}e_{\alpha\beta} + A_\alpha\gamma_\alpha + \bar{A}\gamma_3 - R\theta - \sum_{M=0}^{L}(C^M_i E_{Mi} + F^M_i H_{Mi}),\\
M_{\alpha\beta} &= B_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} + B_{\alpha\beta\lambda}\kappa_{3\lambda} - Q_{\alpha\beta}\phi - \sum_{M=0}^{L}(\bar{C}^M_{\alpha\beta i}E_{Mi} + \bar{F}^M_{\alpha\beta i}H_{Mi}),\\
M_{3\alpha} &= B_{\lambda\mu\alpha}\kappa_{\lambda\mu} + B_{\alpha\beta}\kappa_{3\beta} - Q_\alpha\phi - \sum_{M=0}^{L}(\bar{C}^M_{\alpha i}E_{Mi} + \bar{F}^M_{\alpha i}H_{Mi}),\\
\rho\eta &= P_{\alpha\beta}e_{\alpha\beta} + P_\alpha\gamma_\alpha + R\gamma_3 + P\theta - \sum_{M=0}^{L}(R^M_i E_{Mi} + S^M_i H_{Mi}),\\
\rho\eta_1 &= Q_{\alpha\beta}\kappa_{\alpha\beta} + Q_\alpha\kappa_{3\alpha} + Q\phi - \sum_{M=0}^{L}(\bar{R}^M_i E_{Mi} + \bar{S}^M_i H_{Mi}),\\
\bar{D}_{Mi} &= C^M_{\alpha\beta i}e_{\alpha\beta} + C^M_{\alpha i}\gamma_\alpha + C^M_i\gamma_3 + \bar{C}^M_{\alpha\beta i}\kappa_{\alpha\beta} + \bar{C}^M_{\alpha i}\kappa_{3\alpha} - R^M_i\theta - \bar{R}^M_i\phi\\
&\quad + \sum_{M=0}^{L}(L^{MN}_{ij}E_{Nj} + M^{MN}_{ij}H_{Nj}),\\
\bar{B}_{Mi} &= F^M_{\alpha\beta i}e_{\alpha\beta} + F^M_{\alpha i}\gamma_\alpha + F^M_i\gamma_3 + \bar{F}^M_{\alpha\beta i}\kappa_{\alpha\beta} + \bar{F}^M_{\alpha i}\kappa_{3\alpha} - S^M_i\theta - \bar{S}^M_i\phi\\
&\quad + \sum_{M=0}^{L}(M^{NM}_{ji}E_{Nj} + N^{MN}_{ij}H_{Nj}).
\end{aligned}\right\} \quad (6.18)$$

We consider constitutive equations for $\boldsymbol{p}$, $\boldsymbol{p}_1$, ξ, ξ_1, $\boldsymbol{J}_M$ later. The coefficients in (6.16)–(6.18) are to be selected so that the Cosserat theory represents the main properties of a thin homogeneous anisotropic, thermoelastic, magnetic, polarized plate of constant thickness h. To provide motivation for our choice of coefficients, we compare some exact solutions of the equations of the present section with exact solutions of the corresponding problems in the three-dimensional theory. This enables us to express the coefficients in (6.16)–(6.18) in terms of the three-dimensional coefficients given in Appendix D. Of course, these coefficients, in their turn, must be evaluated by suitable comparisons with experiment. This presents some difficulties in the general anisotropic case owing to the large number of constants that describe the theory. However, in many problems of interest, the body has material symmetries, which enable us to reduce the number of constants to be identified.

It is clear from Appendix B that there are a number of ways in which the constitutive results from the theory of Cosserat surfaces may be related to those that can be obtained in the context of three-dimensional theory. Here we relate the two either by (*a*) polynomial and Legendre function representations of the thermomechanical and electromagnetic three-dimensional fields, respectively; or by (*b*) polynomial and harmonic function representations of these fields, respectively. In what follows, we refer to these alternatives as cases (*a*) and (*b*), respectively; and corresponding to these cases we get conditions (5.3*a*, *b*) when we impose condition (iii) of § 5 on ψ. Since we have restricted the thermomechanical part of the theory to one director and two temperatures, we refer to (B 1) of Appendix B and take for *case (a)*

$$\left.\begin{aligned}&\lambda_1(z)=z,\quad \mu_1(z)=z,\quad \lambda_N(z)=\mu_N(z)=0\quad (N\geqslant 2),\\ &\psi_N(z)=\chi_N(z)=\{(2N+1)/h\}^{\frac{1}{2}}P_N(\mu),\quad \mu=2z/h,\quad \chi_N^K=\psi_N^K=c_N^K,\end{aligned}\right\}\tag{6.19}$$

where c_N^K is given by (B 12) with $z_2=-z_1=\frac{1}{2}h$; and for

case (b)

$$\left.\begin{aligned}&\lambda_1(z)=z,\quad \mu_1(z)=z,\quad \lambda_N(z)=\mu_N(z)=0\quad (N\geqslant 2),\\ &\psi_N(z)=(2/h)^{\frac{1}{2}}\sin\{\tfrac{1}{2}N\pi(1+\mu)\},\quad \chi_0(z)=h^{-\frac{1}{2}},\\ &\chi_N(z)=(2/h)^{\frac{1}{2}}\cos\{\tfrac{1}{2}N\pi(1+\mu)\}\quad (N=1,2,\ldots),\quad \mu=2z/h,\\ &\psi_N^K=\chi_N^K=0\quad (K\neq N),\quad \psi_N^K=-\chi_N^K=N\pi h^{-1}\quad (K=N).\end{aligned}\right\}\tag{6.20}$$

The method of evaluation of the coefficients in (6.16)–(6.18) follows lines similar to those used previously for an isotropic thermoelastic plate with suitable extensions to allow for electromagnetic effects. We omit details and just present the final results. It is, however, of help first to rewrite equations (6.16) and (6.18) in a partially inverted form to correspond to equations (D 4) in Appendix D by introducing a partial Gibbs function. Considerable use is then made of the formulae (B 2)–(B 8) and (B 26)–(B 31) in Appendix B. The partial Gibbs function is

$$\left.\begin{aligned}&G=\psi-N_{\alpha\beta}e_{\alpha\beta}-k_\alpha\gamma_\alpha-k_3\gamma_3-M_{\alpha\beta}\kappa_{\alpha\beta}-M_{3\alpha}\kappa_{3\alpha}\\ &\quad=\bar{G}(N_{\alpha\beta},k_\alpha,k_3,M_{\alpha\beta},M_{3\alpha},\theta,\phi,E_{Mi},H_{Mi})\end{aligned}\right.$$

and

$$\left.\begin{aligned}&\gamma_3=-\rho\frac{\partial\bar{G}}{\partial k_3},\quad \gamma_\alpha=-\rho\frac{\partial\bar{G}}{\partial k_\alpha},\quad \kappa_{3\alpha}=-\rho\frac{\partial\bar{G}}{\partial M_{3\alpha}},\quad \kappa_{\alpha\beta}=-\rho\frac{\partial\bar{G}}{\partial M_{\alpha\beta}},\\ &e_{\alpha\beta}=-\tfrac{1}{2}\rho\left(\frac{\partial\bar{G}}{\partial N_{\alpha\beta}}+\frac{\partial\bar{G}}{\partial N_{\beta\alpha}}\right),\quad \eta=-\frac{\partial\bar{G}}{\partial\theta},\quad \eta_1=-\frac{\partial\bar{G}}{\partial\phi},\\ &\bar{D}_{Mi}=-\rho\frac{\partial\bar{G}}{\partial E_{Mi}},\quad B_{Mi}=-\rho\frac{\partial\bar{G}}{\partial H_{Mi}}.\end{aligned}\right\}\tag{6.21}$$

Also,

$$\begin{aligned}\rho\bar{G}=&-\tfrac{1}{2}A^*_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu}-A^*_{\alpha\beta\lambda}N_{\alpha\beta}k_\lambda-\tfrac{1}{2}\bar{A}^*_{\alpha\beta}k_\alpha k_\beta-\tfrac{1}{2}\bar{A}^*k_3^2-A^*_\alpha k_\alpha k_3-A'^*_{\alpha\beta}N_{\alpha\beta}k_3\\ &-\tfrac{1}{2}P^*\theta^2-P^*_{\alpha\beta}N_{\alpha\beta}\theta-P^*_\alpha k_\alpha\theta-R^*k_3\theta-\tfrac{1}{2}B^*_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu}-B^*_{\alpha\beta\lambda}M_{\alpha\beta}M_{3\lambda}\\ &-\tfrac{1}{2}B^*_{\alpha\beta}M_{3\alpha}M_{3\beta}-\tfrac{1}{2}Q^*\phi^2-Q^*_{\alpha\beta}M_{\alpha\beta}\phi-Q^*_\alpha M_{3\alpha}\phi+N_{\alpha\beta}\sum_{M=0}^{L}(C^{*M}_{\alpha\beta i}E_{Mi}+F^{*M}_{\alpha\beta i}H_{Mi})\\ &+k_\alpha\sum_{M=0}^{L}(C^{*M}_{\alpha i}E_{Mi}+F^{*M}_{\alpha i}H_{Mi})+k_3\sum_{M=0}^{L}(C^{*M}_{i}E_{Mi}+F^{*M}_{i}H_{Mi})\\ &+M_{\alpha\beta}\sum_{M=0}^{L}(\bar{C}^{*M}_{\alpha\beta i}E_{Mi}+\bar{F}^{*M}_{\alpha\beta i}H_{Mi})+M_{3\alpha}\sum_{M=0}^{L}(\bar{C}^{*M}_{\alpha i}E_{Mi}+\bar{F}^{*M}_{\alpha i}H_{Mi})\\ &+\theta\sum_{M=0}^{L}(R^{*M}_{i}E_{Mi}+S^{*M}_{i}H_{Mi})+\phi\sum_{M=0}^{L}(\bar{R}^{*M}_{i}E_{Mi}+\bar{S}^{*M}_{i}H_{Mi})\\ &+\sum_{M=0}^{L}\sum_{N=0}^{L}\{\tfrac{1}{2}L^{*MN}_{ij}E_{Mi}E_{Nj}+M^{*MN}_{ij}E_{Mi}H_{Nj}+\tfrac{1}{2}N^{*MN}_{ij}H_{Mi}H_{Nj}\},\end{aligned}\tag{6.22}$$

where

$$\left.\begin{aligned}
&A_{\alpha\beta\lambda\mu}A^*_{\lambda\mu\rho\nu}+A_{\alpha\beta\lambda}A^*_{\rho\nu\lambda}+A'_{\alpha\beta}A'^*_{\rho\nu}=\tfrac{1}{2}(\delta_{\alpha\rho}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\rho}),\\
&A_{\alpha\beta\lambda\mu}A^*_{\lambda\mu\rho}+A_{\alpha\beta\lambda}\bar{A}^*_{\lambda\rho}+A'_{\alpha\beta}A^*_{\rho}=0,\quad A_{\alpha\beta\lambda\mu}A'^*_{\lambda\mu}+A_{\alpha\beta\lambda}A^*_{\lambda}+A'_{\alpha\beta}\bar{A}^*=0,\\
&A_{\lambda\mu\alpha}A^*_{\lambda\mu\beta}+\bar{A}_{\alpha\lambda}\bar{A}^*_{\lambda\beta}+A_{\alpha}A^*_{\beta}=\delta_{\alpha\beta},\quad A_{\lambda\mu\alpha}A'^*_{\lambda\mu}+\bar{A}_{\alpha\lambda}A^*_{\lambda}+A_{\alpha}\bar{A}^*=0,\\
&A'_{\alpha\beta}A'^*_{\alpha\beta}+A_{\alpha}A^*_{\alpha}+\bar{A}\bar{A}^*=1,\quad A_{\alpha\beta\lambda\mu}P^*_{\lambda\mu}+A_{\alpha\beta\lambda}P^*_{\lambda}+A'_{\alpha\beta}R^*-P_{\alpha\beta}=0,\\
&A_{\lambda\mu\alpha}P^*_{\lambda\mu}+\bar{A}_{\alpha\lambda}P^*_{\lambda}+A_{\alpha}R^*-P_{\alpha}=0,\quad A'_{\alpha\beta}P^*_{\alpha\beta}+A_{\alpha}P^*_{\alpha}+\bar{A}R^*-R=0,\\
&P-P^*+P_{\alpha\beta}P^*_{\alpha\beta}+P_{\alpha}P^*_{\alpha}+RR^*=0,\quad A_{\alpha\beta\lambda\mu}C^{*M}_{\lambda\mu i}+A_{\alpha\beta\lambda}C^{*M}_{\lambda i}+A'_{\alpha\beta}C^{*M}_{i}+C^{M}_{\alpha\beta i}=0,\\
&A_{\lambda\mu\alpha}C^{*M}_{\lambda\mu i}+\bar{A}_{\alpha\lambda}C^{*M}_{\lambda i}+A_{\alpha}C^{*M}_{i}+C^{M}_{\alpha i}=0,\quad A'_{\alpha\beta}C^{*M}_{\alpha\beta i}+A_{\alpha}C^{*M}_{\alpha i}+\bar{A}C^{*M}_{i}+C^{M}_{i}=0,\\
&A_{\alpha\beta\lambda\mu}F^{*M}_{\lambda\mu i}+A_{\alpha\beta\lambda}F^{*M}_{\lambda i}+A'_{\alpha\beta}F^{*M}_{i}+F^{M}_{\alpha\beta i}=0,\\
&A_{\lambda\mu\alpha}F^{*M}_{\lambda\mu i}+\bar{A}_{\alpha\lambda}F^{*M}_{\lambda i}+A_{\alpha}F^{*M}_{i}+F^{M}_{\alpha i}=0,\quad A'_{\alpha\beta}F^{*M}_{\alpha\beta i}+A_{\alpha}F^{*M}_{\alpha i}+\bar{A}F^{*M}_{i}+F^{M}_{i}=0,\\
&R^{*M}_{i}=R^{M}_{i}-P^*_{\alpha\beta}C^{M}_{\alpha\beta i}-P^*_{\alpha}C^{M}_{\alpha i}-R^*C^{M}_{i},\quad S^{*M}_{i}=S^{M}_{i}-P^*_{\alpha\beta}F^{M}_{\alpha\beta i}-P^*_{\alpha}F^{M}_{\alpha i}-R^*F^{M}_{i},\\
&L^{*MN}_{ij}+L^{MN}_{ij}=C^{*M}_{\alpha\beta i}C^{N}_{\alpha\beta j}+C^{*M}_{\alpha i}C^{N}_{\alpha j}+C^{*M}_{i}C^{N}_{j}+\bar{C}^{*M}_{\alpha\beta i}\bar{C}^{N}_{\alpha\beta j}+\bar{C}^{*M}_{\alpha i}\bar{C}^{N}_{\alpha j},\\
&M^{*MN}_{ij}+M^{MN}_{ij}=C^{*M}_{\alpha\beta i}F^{N}_{\alpha\beta j}+C^{*M}_{\alpha i}F^{N}_{\alpha j}+C^{*M}_{i}F^{N}_{j}+\bar{C}^{*M}_{\alpha\beta i}\bar{F}^{N}_{\alpha\beta j}+\bar{C}^{*M}_{\alpha i}\bar{F}^{N}_{\alpha j},\\
&N^{*MN}_{ij}+N^{MN}_{ij}=F^{*M}_{\alpha\beta i}F^{N}_{\alpha\beta j}+F^{*M}_{\alpha i}F^{N}_{\alpha j}+F^{*M}_{i}F^{N}_{j}+\bar{F}^{*M}_{\alpha\beta i}\bar{F}^{N}_{\alpha\beta j}+\bar{F}^{*M}_{\alpha i}\bar{F}^{N}_{\alpha j},
\end{aligned}\right\}\quad(6.23)$$

with similar formulae in which starred and unstarred symbols are interchanged. Also,

$$\left.\begin{aligned}
&B_{\alpha\beta\lambda\mu}B^*_{\lambda\mu\rho\nu}+B_{\alpha\beta\lambda}B^*_{\rho\nu\lambda}=\tfrac{1}{2}(\delta_{\alpha\rho}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\rho}),\quad B_{\alpha\beta\lambda\mu}B^*_{\lambda\mu\rho}+B_{\alpha\beta\lambda}B^*_{\lambda\rho}=0,\\
&B_{\lambda\mu\alpha}B^*_{\lambda\mu\beta}+B_{\alpha\lambda}B^*_{\lambda\beta}=\delta_{\alpha\beta},\quad B_{\alpha\beta\lambda\mu}Q^*_{\lambda\mu}+B_{\alpha\beta\lambda}Q^*_{\lambda}-Q_{\alpha\beta}=0,\\
&B_{\lambda\mu\alpha}Q^*_{\lambda\mu}+B_{\alpha\lambda}Q^*_{\lambda}-Q_{\alpha}=0,\quad Q-Q^*+Q_{\alpha\beta}Q^*_{\alpha\beta}+Q_{\alpha}Q^*_{\alpha}=0,\\
&B_{\alpha\beta\lambda\mu}\bar{C}^{*M}_{\lambda\mu i}+B_{\alpha\beta\lambda}\bar{C}^{*M}_{\lambda i}+\bar{C}^{M}_{\alpha\beta i}=0,\quad B_{\lambda\mu\alpha}\bar{C}^{*M}_{\lambda\mu i}+B_{\alpha\lambda}\bar{C}^{*M}_{\lambda i}+\bar{C}^{M}_{\alpha i}=0,\\
&B_{\alpha\beta\lambda\mu}\bar{F}^{*M}_{\lambda\mu i}+B_{\alpha\beta\lambda}\bar{F}^{*M}_{\lambda i}+\bar{F}^{M}_{\alpha\beta i}=0,\quad B_{\lambda\mu\alpha}\bar{F}^{*M}_{\lambda\mu i}+B_{\alpha\lambda}\bar{F}^{*M}_{\lambda i}+\bar{F}^{M}_{\alpha i}=0,\\
&\bar{R}^{*M}_{i}=\bar{R}^{M}_{i}-Q^*_{\alpha\beta}\bar{C}^{M}_{\alpha\beta i}-Q^*_{\alpha}\bar{C}^{M}_{\alpha i},\quad \bar{S}^{*M}_{i}=\bar{S}^{M}_{i}-Q^*_{\alpha\beta}\bar{F}^{M}_{\alpha\beta i}-Q^*_{\alpha}\bar{F}^{M}_{\alpha i},
\end{aligned}\right\}\quad(6.24)$$

with similar formulae in which starred and unstarred symbols are interchanged.

We specify values for constitutive coefficients according to the two cases (6.19) and (6.20). First, using (6.19), we have

case (*a*)

$$\left.\begin{aligned}
&\rho=\rho^*h,\quad \rho y^{11}=\tfrac{1}{12}\rho^*h^3,\\
&A^*_{\alpha\beta\lambda\mu}=h^{-1}s_{\alpha\beta\lambda\mu},\quad A^*_{\alpha\beta\lambda}=2h^{-1}s_{\alpha\beta\lambda3},\\
&A'^*_{\alpha\beta}=h^{-1}s_{\alpha\beta33},\quad A^*_{\alpha}=2h^{-1}s_{\alpha333},\quad \bar{A}^*=h^{-1}s_{3333},\\
&P^*_{\alpha\beta}=s_{\alpha\beta},\quad P^*_{\alpha}=2s_{\alpha3},\quad R^*=s_{33},\quad P^*=hc^*,\\
&B^*_{(\alpha\beta)(\lambda\mu)}=12h^{-3}s_{\alpha\beta\lambda\mu},\quad B^*_{(\lambda\mu)\alpha}=24h^{-3}s_{\alpha\beta\lambda3},\quad \beta^{-1}=120/7.\\
&B^*_{\alpha\beta}=4\beta^{-1}h^{-3}s_{\alpha3\beta3},\quad B_{\alpha\beta\lambda\mu}=B_{\beta\alpha\lambda\mu}=B_{\alpha\beta\mu\lambda},\quad B_{\alpha\beta\lambda}=B_{\beta\alpha\lambda},\\
&Q^*_{(\alpha\beta)}=s_{\alpha\beta},\quad Q^*_{\alpha}=2s_{\alpha3},\quad Q^*=\tfrac{1}{12}h^3c^*,\quad Q_{\alpha\beta}=Q_{\beta\alpha},
\end{aligned}\right\}\quad(6.25)$$

and

$$\left.\begin{aligned}
&C^{*0}_{\alpha\beta i} = h^{-\frac{1}{2}}k^{*}_{\alpha\beta i}, \quad C^{*0}_{\alpha i} = 2h^{-\frac{1}{2}}k^{*}_{\alpha 3i}, \quad C^{*0}_{i} = h^{-\frac{1}{2}}k^{*}_{33i},\\
&F^{*0}_{\alpha\beta i} = h^{-\frac{1}{2}}l^{*}_{\alpha\beta i}, \quad F^{*0}_{\alpha i} = 2h^{-\frac{1}{2}}l^{*}_{\alpha 3i}, \quad F^{*0}_{i} = h^{-\frac{1}{2}}l^{*}_{33i},\\
&R^{*0}_{i} = h^{\frac{1}{2}}f^{*}_{i}, \quad S^{*0}_{i} = h^{\frac{1}{2}}g^{*}_{i},\\
&\bar{C}^{*1}_{(\alpha\beta)i} = c^{0}_{1}h^{-\frac{1}{2}}k^{*}_{\alpha\beta i}, \quad \bar{C}^{*1}_{\alpha i} = 2c^{0}_{1}h^{-\frac{1}{2}}k^{*}_{\alpha 3i}, \quad \bar{C}^{1}_{\alpha\beta i} = \bar{C}^{1}_{\beta\alpha i},\\
&\bar{F}^{*1}_{(\alpha\beta)i} = c^{0}_{1}h^{-\frac{1}{2}}l^{*}_{\alpha\beta i}, \quad \bar{F}^{*1}_{\alpha i} = 2c^{0}_{1}h^{-\frac{1}{2}}l^{*}_{\alpha 3i}, \quad \bar{F}^{1}_{\alpha\beta i} = \bar{F}^{1}_{\beta\alpha i},\\
&\bar{R}^{*1}_{i} = h^{\frac{1}{2}}f^{*}_{i}/c^{0}_{1}, \quad \bar{S}^{*1}_{i} = h^{\frac{1}{2}}g^{*}_{i}/c^{0}_{1},\\
&L^{*MM}_{ij} = f^{*}_{ij}, \quad M^{*MM}_{ij} = h^{*}_{ij}, \quad N^{*MM}_{ij} = g^{*}_{ij},
\end{aligned}\right\} \tag{6.26}$$

with

$$\left.\begin{aligned}
&\left.\begin{aligned}
&C^{*M}_{\alpha\beta i} = 0, \quad C^{*M}_{\alpha i} = 0, \quad C^{*M}_{i} = 0,\\
&F^{*M}_{\alpha\beta i} = 0, \quad F^{*M}_{\alpha i} = 0, \quad F^{*M}_{i} = 0,\\
&R^{*M}_{i} = 0, \quad S^{*M}_{i} = 0
\end{aligned}\right\} \quad (M = 1, 2, \ldots, L)\\
&\left.\begin{aligned}
&\bar{C}^{*M}_{\alpha\beta i} = 0, \quad \bar{C}^{*M}_{\alpha i} = 0, \quad \bar{C}^{*M}_{i} = 0,\\
&\bar{F}^{*M}_{\alpha\beta i} = 0, \quad \bar{F}^{*M}_{\alpha i} = 0, \quad \bar{F}^{*M}_{i} = 0,\\
&\bar{R}^{*M}_{i} = 0, \quad \bar{S}^{*M}_{i} = 0
\end{aligned}\right\} \quad (M = 0, 2, \ldots, L)\\
&L^{*MN}_{ij} = M^{*MN}_{ij} = N^{*MN}_{ij} = 0 \qquad (M \neq N).
\end{aligned}\right\} \tag{6.27}$$

In addition, the coefficients $\bar{A}_{\alpha\beta}$ are chosen so that the thickness shear frequencies obtained from the two-dimensional equations agree with the corresponding ones obtained from the three-dimensional equations. For a general anisotropic plate, it is difficult to give analytical expressions for $\bar{A}_{\alpha\beta}$.

In discussing constitutive equations for the functions $\boldsymbol{p}$, $\boldsymbol{p}_1$, ξ, ξ_1, $\boldsymbol{J}_M$ by the direct theory for case (a), we need restrictions, which arise from further thermodynamical considerations. The constitutive coefficients in these equations are then expressed in terms of the three-dimensional coefficients in (D 1) with the help of results in Appendix B. Here we make direct use of (D 1), (D 2) and Appendix B and list the final results:

case (a)

$$\left.\begin{aligned}
&p_\alpha = -hk_{\alpha\beta}\theta_{,\beta} - hk_{\alpha 3}\phi - \bar{a}_{\alpha i}h^{\frac{1}{2}}E_{0i}, \quad p_{1\alpha} = -\tfrac{1}{12}h^3k_{\alpha\beta}\phi_{,\beta} - (h^{\frac{1}{2}}/c^0_1)\,\bar{a}_{\alpha i}E_{1i},\\
&\xi = 0, \quad \rho\xi_1 = -hk_{3\alpha}\theta_{,\alpha} - hk_{33}\phi - h^{\frac{1}{2}}\bar{a}_{3i}E_{0i}, \quad J_{0i} = h^{\frac{1}{2}}(l_{i\alpha}\theta_{,\alpha} + l_{i3}\phi) + b_{ij}E_{0j},\\
&J_{1i} = (h^{\frac{1}{2}}/c^0_1)\,l_{i\alpha}\phi_{,\alpha} + b_{ij}E_{ij}, \quad J_{Mi} = b_{ij}E_{Mj} \quad (M = 2, 3, \ldots, L).
\end{aligned}\right\} \tag{6.28}$$

The foregoing results for a plate with general anisotropic properties often simplify considerably when the material has special symmetries.

When the Cosserat plate theory is interpreted under the conditions of case (b) in (6.20), the material coefficients listed in (6.25) still have the same values; these are the coefficients associated with the thermomechanical part of the theory. The electromagnetic coefficients in (6.26) and (6.27) are replaced by the following equations (6.29):

case (*b*)

$$C^{*0}_{\alpha\beta\lambda} = 0, \quad C^{*0}_{\alpha\beta3} = h^{-\frac{1}{2}}k^{*}_{\alpha\beta3}, \quad C^{*M}_{\alpha\beta\lambda} = \frac{2^{\frac{1}{2}}}{M\pi h^{\frac{1}{2}}}\{1-(-1)^{M}\}\,k^{*}_{\alpha\beta\lambda}, \quad C^{*M}_{\alpha\beta3} = 0 \quad (M \geqslant 1),$$

$$C^{*0}_{\alpha\lambda} = 0, \quad C^{*0}_{\alpha3} = 2h^{-\frac{1}{2}}k^{*}_{\alpha33}, \quad C^{*M}_{\alpha\lambda} = \frac{2^{\frac{3}{2}}}{M\pi h^{\frac{1}{2}}}\{1-(-1)^{M}\}\,k^{*}_{\alpha3\lambda}, \quad C^{*M}_{\alpha3} = 0 \quad (M \geqslant 1),$$

$$C^{*0}_{\lambda} = 0, \quad C^{*0}_{3} = h^{-\frac{1}{2}}k^{*}_{333}, \quad C^{*M}_{\lambda} = \frac{2^{\frac{1}{2}}}{M\pi h^{\frac{1}{2}}}\{1-(-1)^{M}\}\,k^{*}_{33\lambda}, \quad C^{*M}_{3} = 0 \quad (M \geqslant 1),$$

$$F^{*0}_{\alpha\beta\lambda} = h^{-\frac{1}{2}}l^{*}_{\alpha\beta\lambda}, \quad F^{*0}_{\alpha\beta3} = 0, \quad F^{*M}_{\alpha\beta\lambda} = 0, \quad F^{*M}_{\alpha\beta3} = \frac{2^{\frac{1}{2}}}{M\pi h^{\frac{1}{2}}}\{1-(-1)^{M}\}\,l^{*}_{\alpha\beta3} \quad (M \geqslant 1),$$

$$F^{*0}_{\alpha\lambda} = 2h^{-\frac{1}{2}}l^{*}_{\alpha3\lambda}, \quad F^{*0}_{\alpha3} = 0, \quad F^{*M}_{\alpha\lambda} = 0, \quad F^{*M}_{\alpha3} = \frac{2^{\frac{3}{2}}}{M\pi h^{\frac{1}{2}}}\{1-(-1)^{M}\}\,l^{*}_{\alpha33} \quad (M \geqslant 1),$$

$$F^{*0}_{\lambda} = h^{-\frac{1}{2}}l^{*}_{33\lambda}, \quad F^{*0}_{3} = 0, \quad F^{*M}_{\lambda} = 0, \quad F^{*M}_{3} = \frac{2^{\frac{1}{2}}}{M\pi h^{\frac{1}{2}}}\{1-(-1)^{M}\}\,l^{*}_{333} \quad (M \geqslant 1),$$

$$\bar{C}^{*0}_{\alpha\beta\lambda} = 0, \quad \bar{C}^{*0}_{\alpha\beta3} = 0, \quad \bar{C}^{*M}_{(\alpha\beta)\lambda} = -\frac{12}{2^{\frac{1}{2}}h^{\frac{3}{2}}\pi M}\{1+(-1)^{M}\}\,k^{*}_{\alpha\beta\lambda},$$

$$\bar{C}^{*M}_{(\alpha\beta)3} = -\frac{24}{2^{\frac{1}{2}}h^{\frac{3}{2}}\pi^{2}M^{2}}\{1-(-1)^{M}\}\,k^{*}_{\alpha\beta3} \quad (M \geqslant 1), \quad \bar{C}^{*0}_{\alpha\lambda} = 0, \quad \bar{C}^{*0}_{\alpha3} = 0, \quad \bar{C}^{M}_{\alpha\beta i} = \bar{C}^{M}_{\beta\alpha i},$$

$$\bar{C}^{*M}_{\alpha\lambda} = -\frac{24}{2^{\frac{1}{2}}h^{\frac{3}{2}}\pi M}\{1+(-1)^{M}\}\,k^{*}_{\alpha3\lambda}, \quad \bar{C}^{*M}_{\alpha3} = -\frac{48}{2^{\frac{1}{2}}h^{\frac{3}{2}}\pi^{2}M^{2}}\{1-(-1)^{M}\}\,k^{*}_{\alpha33} \quad (M \geqslant 1),$$

$$\bar{F}^{*0}_{\alpha\beta\lambda} = 0, \quad \bar{F}^{*0}_{\alpha\beta3} = 0, \quad \bar{F}^{M}_{\alpha\beta i} = \bar{F}^{M}_{\beta\alpha i}, \quad \bar{F}^{*M}_{(\alpha\beta)\lambda} = -\frac{24}{2^{\frac{1}{2}}h^{\frac{3}{2}}\pi^{2}M^{2}}\{1-(-1)^{M}\}\,l^{*}_{\alpha\beta\lambda},$$

$$\bar{F}^{*M}_{(\alpha\beta)3} = -\frac{12}{2^{\frac{1}{2}}h^{\frac{3}{2}}\pi M}\{1+(-1)^{M}\}\,l^{*}_{\alpha\beta3} \quad (M \geqslant 1), \quad \bar{F}^{*0}_{\alpha\lambda} = 0, \quad \bar{F}^{*0}_{\alpha3} = 0,$$

$$\bar{F}^{*M}_{\alpha\lambda} = -\frac{48}{2^{\frac{1}{2}}h^{\frac{3}{2}}\pi^{2}M^{2}}\{1-(-1)^{M}\}\,l^{*}_{\alpha3\lambda}, \quad \bar{F}^{*M}_{\alpha3} = -\frac{24}{2^{\frac{1}{2}}h^{\frac{3}{2}}\pi M}\{1+(-1)^{M}\}\,l^{*}_{\alpha33} \quad (M \geqslant 1),$$

$$R^{*0}_{\alpha} = 0, \quad R^{*0}_{3} = h^{\frac{1}{2}}f^{*}_{3}, \quad R^{*M}_{\alpha} = \frac{2^{\frac{1}{2}}h^{\frac{1}{2}}}{M\pi}\{1-(-1)^{M}\}\,f^{*}_{\alpha}, \quad R^{*M}_{3} = 0 \quad (M \geqslant 1),$$

$$S^{*0}_{\alpha} = h^{\frac{1}{2}}g^{*}_{\alpha}, \quad S^{*0}_{3} = 0, \quad S^{*M}_{3} = 0, \quad S^{*M}_{3} = \frac{2^{\frac{1}{2}}h^{\frac{1}{2}}}{M\pi}\{1-(-1)^{M}\}\,g^{*}_{3} \quad (M \geqslant 1),$$

$$\bar{R}^{*0}_{\alpha} = 0, \quad \bar{R}^{*0}_{3} = 0, \quad \bar{R}^{*M}_{\alpha} = -\frac{h^{\frac{3}{2}}}{2^{\frac{1}{2}}M\pi}\{1+(-1)^{M}\}\,f^{*}_{\alpha},$$

$$\bar{R}^{*M}_{3} = -\frac{2^{\frac{1}{2}}h^{\frac{3}{2}}}{M^{2}\pi^{2}}\{1-(-1)^{M}\}\,f^{*}_{3} \quad (M \geqslant 1), \quad \bar{S}^{*0}_{\alpha} = 0, \quad \bar{S}^{*0}_{3} = 0,$$

$$\bar{S}^{*M}_{\alpha} = -\frac{2^{\frac{1}{2}}h^{\frac{3}{2}}}{M^{2}\pi^{2}}\{1-(-1)^{M}\}\,g^{*}_{\alpha}, \quad \bar{S}^{*M}_{3} = -\frac{h^{\frac{3}{2}}}{2^{\frac{1}{2}}M\pi}\{1+(-1)^{M}\}\,g^{*}_{3} \quad (M \geqslant 1), \quad L^{*M0}_{\alpha\beta} = 0,$$

$$L^{*0N}_{\alpha\beta} = 0 \quad (M, N = 0, 1, \ldots), \quad L^{*MN}_{\alpha\beta} = f^{*}_{\alpha\beta}\,\delta_{MN} \quad (M, N \geqslant 1), \quad L^{*0N}_{\alpha3} = 0 \quad (N = 0, 1, \ldots),$$

$$L^{*MM}_{\alpha3} = 0 \quad (M = 0, 1, \ldots), \quad L^{*M0}_{\alpha3} = \frac{2^{\frac{1}{2}}}{\pi M}\{1-(-1)^{M}\}\,f^{*}_{\alpha3} \quad (M \geqslant 1),$$

$$L^{*MN}_{\alpha3} = \frac{2M}{\pi}\left\{\frac{1-(-1)^{M+N}}{M^{2}-N^{2}}\right\}f^{*}_{\alpha3} \quad (M \neq N, M, N \geqslant 1), \quad L^{*MN}_{33} = f^{*}_{33}\,\delta_{MN} \quad (M, N = 0, 1, \ldots),$$

$$M^{*0N}_{\alpha\beta} = 0 \quad (N = 0, 1, \ldots), \quad M^{*MM}_{\alpha\beta} = 0 \quad (M = 0, 1, \ldots),$$

$$M^{*M0}_{\alpha\beta} = \frac{2^{\frac{1}{2}}}{\pi M}\{1-(-1)^{M}\}\,h^{*}_{\alpha\beta} \quad (M \geqslant 1),$$

$$M^{*MN}_{\alpha\beta} = \frac{2M}{\pi}\left\{\frac{1-(-1)^{M+N}}{M^{2}-N^{2}}\right\}h^{*}_{\alpha\beta} \quad (M \neq N, M, N \geqslant 1),$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$$
\begin{aligned}
&M^{*0N}_{\alpha 3} = M^{*M0}_{\alpha 3} = 0 \quad (M, N = 0, 1, \ldots), \quad M^{*MN}_{\alpha 3} = h^{*}_{\alpha 3}\,\delta_{MN} \quad (M, N \geqslant 1),\\
&M^{*MN}_{3\alpha} = h^{*}_{3\alpha}\,\delta_{MN} \quad (M, N \geqslant 0), \quad M^{*M0}_{33} = 0 \quad (M = 0, 1, \ldots),\\
&M^{*MM}_{33} = 0 \quad (M = 0, 1, \ldots), \quad M^{*0N}_{33} = \frac{2^{\frac{1}{2}}}{\pi N}\{1-(-1)^{N}\}h^{*}_{33} \quad (N \geqslant 1),\\
&M^{*MN}_{33} = \frac{2N}{\pi}\left\{\frac{1-(-1)^{M+N}}{N^2-M^2}\right\}h^{*}_{33} \quad (M \neq N, M, N \geqslant 1), \quad N^{*MN}_{\alpha\beta} = g^{*}_{\alpha\beta}\,\delta_{MN} \quad (M, N = 0, 1, \ldots),\\
&N^{*M0}_{\alpha 3} = 0 \quad (M = 0, 1, \ldots), \quad N^{*MM}_{\alpha 3} = 0 \quad (M = 0, 1, \ldots),\\
&N^{*0N}_{\alpha 3} = \frac{2^{\frac{1}{2}}}{\pi N}\{1-(-1)^{N}\}g^{*}_{\alpha 3} \quad (N \geqslant 1),\\
&N^{*MN}_{\alpha 3} = \frac{2N}{\pi}\left\{\frac{1-(-1)^{M+N}}{N^2-M^2}\right\}g^{*}_{\alpha 3} \quad (M \neq N, M, N \geqslant 1),\\
&N^{*M0}_{33} = 0, \quad N^{*0N}_{33} = 0 \quad (M, N = 0, 1, \ldots), \quad N^{*MN}_{33} = g^{*}_{33}\,\delta_{MN} \quad (M, N \geqslant 1).
\end{aligned} \tag{6.29}
$$

7. Partially restricted theory of a Cosserat surface

For many purposes it is useful to develop a restricted (or a constrained) theory of a Cosserat surface, which results in somewhat simpler field equations as far as the mechanical variables are concerned. In such a theory the director $\boldsymbol{d} = \boldsymbol{d}_1$ at each point of the surface is constrained so that its component along the normal $\boldsymbol{A}_3$ to the material surface of $\mathscr{C}$ in the reference configuration is always constant and $\boldsymbol{d}_N = \boldsymbol{0}$ for $N \geqslant 2$. We omit details of such a partially restricted theory for the general nonlinear case but only quote results for the linear elastic Cosserat plate as a restricted case of the results of § 6. Thus we have

$$\delta_3 = 0, \quad \kappa_{3\alpha} = \delta_{3,\alpha} = 0$$

and k_3, $M_{3\alpha}$ become arbitrary functions of θ^α, t, not determined by constitutive equations. Instead of (6.15) we now obtain

$$
\left.
\begin{aligned}
&\psi = \tilde{\psi}(e_{\alpha\beta}, \gamma_\alpha, \kappa_{\alpha\beta}, \theta, \phi, E_{Mi}, H_{Mi}), \quad k_\alpha = \rho\frac{\partial\tilde{\psi}}{\partial\gamma_\alpha}, \quad M_{\alpha\beta} = \rho\frac{\partial\tilde{\psi}}{\partial\kappa_{\alpha\beta}},\\
&N_{\alpha\beta} = \tfrac{1}{2}\rho\left(\frac{\partial\tilde{\psi}}{\partial e_{\alpha\beta}} + \frac{\partial\tilde{\psi}}{\partial e_{\beta\alpha}}\right), \quad \eta = -\frac{\partial\tilde{\psi}}{\partial\theta}, \quad \eta_1 = -\frac{\partial\tilde{\psi}}{\partial\phi}, \quad \bar{D}_{Mi} = -\rho\frac{\partial\tilde{\psi}}{\partial E_{Mi}}, \quad B_{Mi} = -\rho\frac{\partial\tilde{\psi}}{\partial H_{Mi}},
\end{aligned}
\right\} \tag{7.1}
$$

with corresponding equations of motion

$$\rho\ddot{u}_i = \rho f_i + N_{i\alpha,\alpha}, \quad N_{\alpha\beta} = N_{\beta\alpha}, \quad \rho y^{11}\ddot{\delta}_\beta = \rho l_\beta - k_\beta + M_{\beta\alpha,\alpha}, \quad k_\alpha = N_{3\alpha}, \quad M_{3\alpha,\alpha} + \rho l_3 - k_3 = 0. \tag{7.2}$$

We take $l_3 = 0$ and adopt the special solution $M_{3\alpha} = 0$, $k_3 = 0$ throughout this paper. The entropy balance and electromagnetic equations are the same as in (6.8)–(6.12). The expression for $\tilde{\psi}$ in (6.16) is replaced by†

$$
\begin{aligned}
\rho\tilde{\psi} = {} & \tfrac{1}{2}A_{\alpha\beta\lambda\mu}e_{\alpha\beta}e_{\lambda\mu} + A_{\alpha\beta\lambda}e_{\alpha\beta}\gamma_\lambda + \tfrac{1}{2}\bar{A}_{\alpha\beta}\gamma_\alpha\gamma_\beta + \tfrac{1}{2}B_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu} - \tfrac{1}{2}P\theta^2 - P_{\alpha\beta}e_{\alpha\beta}\theta - P_\alpha\gamma_\alpha\theta\\
& - \tfrac{1}{2}Q\phi^2 - Q_{\alpha\beta}\kappa_{\alpha\beta}\phi - e_{\alpha\beta}\sum_{M=0}^{L}(C^M_{\alpha\beta i}E_{Mi} + F^M_{\alpha\beta i}H_{Mi}) - \gamma_\alpha\sum_{M=0}^{L}(C^M_{\alpha i}E_{Mi} + F^M_{\alpha i}H_{Mi})\\
& - \kappa_{\alpha\beta}\sum_{M=0}^{L}(\bar{C}^M_{\alpha\beta i}E_{Mi} + \bar{F}^M_{\alpha\beta i}H_{Mi}) + \theta\sum_{M=0}^{L}(R^M_i E_{Mi} + S^M_i H_{Mi}) + \phi\sum_{M=0}^{L}(\bar{R}^M_i E_{Mi} + \bar{S}^M_i H_{Mi})\\
& - \sum_{M=0}^{L}\sum_{N=0}^{L}(\tfrac{1}{2}L^{MN}_{ij}E_{Mi}E_{Nj} + M^{MN}_{ij}E_{Mi}H_{Nj} + \tfrac{1}{2}N^{MN}_{ij}H_{Mi}H_{Nj}).
\end{aligned} \tag{7.3}
$$

† To avoid notational difficulties, the same symbols are used here as in § 6 but they now have different values.

The constant coefficients in (7.3) have the symmetries recorded in (6.17). From (7.2) and (7.3) it follows that

$$\left.\begin{aligned}
N_{\alpha\beta} &= N_{\beta\alpha} = A_{\alpha\beta\lambda\mu}e_{\lambda\mu}+A_{\alpha\beta\lambda}\gamma_\lambda-P_{\alpha\beta}\theta-\sum_{M=0}^{L}(C^M_{\alpha\beta i}E_{Mi}+F^M_{\alpha\beta i}H_{Mi}),\\
k_\alpha &= A_{\lambda\mu\alpha}e_{\lambda\mu}+\bar{A}_{\alpha\beta}\gamma_\beta-P_\alpha\theta-\sum_{M=0}^{L}(C^M_{\alpha i}E_{Mi}+F^M_{\alpha i}H_{Mi}),\\
M_{\alpha\beta} &= B_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}-Q_{\alpha\beta}\phi-\sum_{M=0}^{L}(\bar{C}^M_{\alpha\beta i}E_{Mi}+\bar{F}^M_{\alpha\beta i}H_{Mi}),\\
\rho\eta &= P_{\alpha\beta}e_{\alpha\beta}+P_\alpha\gamma_\alpha+P\theta-\sum_{M=0}^{L}(R^M_iE_{Mi}+S^M_iH_{Mi}),\\
\rho\eta_1 &= Q_{\alpha\beta}\kappa_{\alpha\beta}+Q\phi-\sum_{M=0}^{L}(\bar{R}^M_iE_{Mi}+\bar{S}^M_iH_{Mi}),\\
\bar{D}_{Mi} &= C^M_{\alpha\beta i}e_{\alpha\beta}+C^M_{\alpha i}\gamma_\alpha+\bar{C}^M_{\alpha\beta i}\kappa_{\alpha\beta}-R^M_i\theta-\bar{R}^M_i\phi+\sum_{N=0}^{L}(L^{MN}_{ij}E_{Nj}+M^{MN}_{ij}H_{Nj}),\\
B_{Mi} &= F^M_{\alpha\beta i}e_{\alpha\beta}+F^M_{\alpha i}\gamma_\alpha+\bar{F}^M_{\alpha\beta i}\kappa_{\alpha\beta}-S^M_i\theta-\bar{S}^M_i\phi+\sum_{N=0}^{L}(M^{NM}_{ji}E_{Nj}+N^{MN}_{ij}H_{Nj}),
\end{aligned}\right\} \quad (7.4)$$

with partially inverse relations

$$G = \tilde{G}(N_{\alpha\beta}, k_\alpha, M_{\alpha\beta}, \theta, \phi, E_{Mi}, H_{Mi}),$$

$$\gamma_\alpha = -\rho\frac{\partial\tilde{G}}{\partial k_\alpha},\quad \kappa_{\alpha\beta} = -\rho\frac{\partial\tilde{G}}{\partial M_{\alpha\beta}},\quad e_{\alpha\beta} = -\tfrac{1}{2}\rho\left(\frac{\partial\tilde{G}}{\partial N_{\alpha\beta}}+\frac{\partial\tilde{G}}{\partial N_{\beta\alpha}}\right),$$

$$\eta = -\frac{\partial\tilde{G}}{\partial\theta},\quad \eta_1 = -\frac{\partial\tilde{G}}{\partial\phi},\quad \bar{D}_{Mi} = -\rho\frac{\partial\tilde{G}}{\partial E_{Mi}},\quad B_{Mi} = -\rho\frac{\partial\tilde{G}}{\partial H_{Mi}}$$

where

$$\begin{aligned}
\rho\tilde{G} = {} & -\tfrac{1}{2}A^*_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu}-A^*_{\alpha\beta\lambda}N_{\alpha\beta}k_\lambda-\tfrac{1}{2}\bar{A}^*_{\alpha\beta}k_\alpha k_\beta-\tfrac{1}{2}B^*_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu}-\tfrac{1}{2}P^*\theta\\
& -P^*_{\alpha\beta}N_{\alpha\beta}\theta-P^*_\alpha k_\alpha\theta-\tfrac{1}{2}Q^*\phi^2-Q^*_{\alpha\beta}M_{\alpha\beta}\phi+N_{\alpha\beta}\sum_{M=0}^{L}(C^{*M}_{\alpha\beta i}E_{Mi}+F^{*M}_{\alpha\beta i}H_{Mi})\\
& +k_\alpha\sum_{M=0}^{L}(C^{*M}_{\alpha i}E_{Mi}+F^{*M}_{\alpha i}H_{Mi})+M_{\alpha\beta}\sum_{M=0}^{L}(\bar{C}^{*M}_{\alpha\beta i}E_{Mi}+\bar{F}^{*M}_{\alpha\beta i}H_{Mi})\\
& +\theta\sum_{M=0}^{L}(R^{*M}_iE_{Mi}+S^{*M}_iH_{Mi})+\phi\sum_{M=0}^{L}(\bar{R}^{*M}_iE_{Mi}+\bar{S}^{*M}_iH_{Mi})\\
& +\sum_{M=0}^{L}\sum_{N=0}^{L}(\tfrac{1}{2}L^{*MN}_{ij}E_{Mi}E_{Nj}+M^{*MN}_{ij}E_{Mi}H_{Nj}+\tfrac{1}{2}N^{*MN}_{ij}H_{Mi}H_{Nj})
\end{aligned} \quad (7.5)$$

and

$$\left.\begin{gathered}
A_{\alpha\beta\lambda\mu}A^*_{\lambda\mu\rho\nu}+A_{\alpha\beta\lambda}A^*_{\rho\nu\lambda} = \tfrac{1}{2}(\delta_{\alpha\rho}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\rho}),\quad A_{\alpha\beta\lambda\mu}A^*_{\lambda\mu\rho}+A_{\alpha\beta\lambda}\bar{A}^*_{\lambda\rho} = 0,\\
A_{\lambda\mu\alpha}A^*_{\lambda\mu\beta}+\bar{A}_{\alpha\lambda}\bar{A}^*_{\lambda\beta} = \delta_{\alpha\beta},\quad A_{\alpha\beta\lambda\mu}P^*_{\lambda\mu}+A_{\alpha\beta\lambda}P^*_\lambda-P_{\alpha\beta} = 0,\quad A_{\lambda\mu\alpha}P^*_{\lambda\mu}+\bar{A}_{\alpha\lambda}P^*_\lambda-P_\alpha = 0,\\
P-P^*+P_{\alpha\beta}P^*_{\alpha\beta}+P_\alpha P^*_\alpha = 0,\quad A_{\alpha\beta\lambda\mu}C^{*M}_{\lambda\mu i}+A_{\alpha\beta\lambda}C^{*M}_{\lambda i}+C^M_{\alpha\beta i} = 0,\\
A_{\lambda\mu\alpha}C^{*M}_{\lambda\mu i}+\bar{A}_{\alpha\lambda}C^{*M}_{\lambda i}+C^M_{\alpha i} = 0,\quad A_{\alpha\beta\lambda\mu}F^{*M}_{\lambda\mu i}+A_{\alpha\beta\lambda}F^{*M}_{\lambda i}+F^M_{\alpha\beta i} = 0,\\
A_{\lambda\mu\alpha}F^{*M}_{\lambda\mu i}+\bar{A}_{\alpha\lambda}F^{*M}_{\lambda i}+F^M_{\alpha i} = 0,\quad R^{*M}_i = R^M_i-P^*_{\alpha\beta}C^M_{\alpha\beta i}-P^*_\alpha C^M_{\alpha i},\\
S^{*M}_i = S^M_i-P^*_{\alpha\beta}F^M_{\alpha\beta i}-P^*_\alpha F^M_{\alpha i},\quad L^{*MN}_{ij}+L^{MN}_{ij} = C^{*M}_{\alpha\beta i}C^N_{\alpha\beta j}+C^{*M}_{\alpha i}C^N_{\alpha j}+\bar{C}^{*M}_{\alpha\beta i}\bar{C}^N_{\alpha\beta j},\\
M^{*MN}_{ij}+M^{MN}_{ij} = C^{*M}_{\alpha\beta i}F^N_{\alpha\beta j}+C^{*M}_{\alpha i}F^N_{\alpha j}+\bar{C}^{*M}_{\alpha\beta i}\bar{F}^N_{\alpha\beta j},\\
N^{*MN}_{ij}+N^{MN}_{ij} = F^{*M}_{\alpha\beta i}F^N_{\alpha\beta j}+F^{*M}_{\alpha i}F^N_{\alpha j}+\bar{F}^{*M}_{\alpha\beta i}\bar{F}^N_{\alpha\beta j},
\end{gathered}\right\} \quad (7.6)$$

with similar formulae in which starred and unstarred symbols are interchanged. Also

$$\left.\begin{aligned} &B_{\alpha\beta\lambda\mu}B^*_{\lambda\mu\rho\nu} = \tfrac{1}{2}(\delta_{\alpha\rho}\delta_{\beta\nu}+\delta_{\alpha\nu}\delta_{\beta\rho}), \quad B_{\alpha\beta\lambda\mu}Q^*_{\lambda\mu}-Q_{\alpha\beta} = 0, \\ &Q-Q^*+Q_{\alpha\beta}Q^*_{\alpha\beta} = 0, \quad B_{\alpha\beta\lambda\mu}\bar{C}^{*M}_{\lambda\mu i}+\bar{C}^M_{\alpha\beta i} = 0, \quad B_{\alpha\beta\lambda\mu}\bar{F}^{*M}_{\lambda\mu i}+\bar{F}^M_{\alpha\beta i} = 0, \\ &\bar{R}^{*M}_i = \bar{R}^M_i - Q^*_{\alpha\beta}\bar{C}^M_{\alpha\beta i}, \quad \bar{S}^{*M}_i = \bar{S}^M_i - Q^*_{\alpha\beta}\bar{F}^M_{\alpha\beta i}, \end{aligned}\right\} \tag{7.7}$$

with similar formulae in which starred and unstarred symbols are interchanged.

Values of the coefficients in (7.3) may be found in terms of the three-dimensional coefficients by a procedure similar to that used in § 6. This leads to the values listed in (6.25)–(6.27) for case (a) and in (6.25) and (6.29) for case (b), for the relevant coefficients. Because equations (7.6), (7.7) are different from (6.23), (6.24), which connect the unstarred and starred coefficients, different values are found for the coefficients in (7.3) from those in (6.16). The equations for evaluating $\bar{A}_{\alpha\beta}$ are slightly simpler here than those in § 6. The thickness shear frequencies ω in the present two-dimensional theory are given by the equation

$$\det\left(\tfrac{1}{12}\rho^* h^3\omega^2\delta_{\alpha\beta}-\bar{A}_{\alpha\beta}\right) = 0. \tag{7.8}$$

These frequencies are to be equated to the two lowest roots of the frequency equation

$$\det\left(\rho^*\pi^{-2}h^2\omega^2\delta_{ij}-c_{i3j3}\right) = 0. \tag{7.9}$$

Mindlin (1961) has used the same comparison as a basis for finding correction factors, which he inserts into his theory. As an illustration he considers the case when there is symmetry under rotation about the $\boldsymbol{e}_2$-direction through an angle π. Then, recalling (D 7) in Appendix D, from (7.8) and (7.9) we obtain

$$\left.\begin{aligned} &\bar{A}_{12} = 0, \quad \bar{A}_{22} = \tfrac{1}{12}\pi^2 h c_{2323}, \\ &\bar{A}_{11} = \tfrac{1}{24}\pi^2 h[c_{1313}+c_{3333}-\{(c_{3333}-c_{1313})^2+4c^2_{1333}\}^{\frac{1}{2}}]. \end{aligned}\right\} \tag{7.10}$$

The constitutive equations for $p_\alpha, p_{1\alpha}, \xi, \xi_1, J_{Mi}$ are still given by (6.28) for case (a).

8. A restricted theory of shells

In application of the thermomechanical theory of shells, it is often sufficient to use a restricted theory in which a director either is not admitted or is assumed to be coincident with the outward unit normal $\boldsymbol{a}_3$. A direct thermomechanical theory of this kind (corresponding to the classical Kirchhoff–Love theory of shells) has been constructed by Naghdi (1972, §§ 10, 15), who also refers to previous approximate nonlinear theories of this type developed by approximations from the three-dimensional equations. Here, we follow the procedure of Naghdi (1972) but include electromagnetic effects and allow also for temperature variation across the shell thickness. We begin with the kinematical results in (B 18), namely

$$\left.\begin{aligned} &\boldsymbol{F} = \boldsymbol{a}_i\otimes\boldsymbol{A}^i, \quad \boldsymbol{a}_i = \boldsymbol{F}\boldsymbol{A}_i, \quad \boldsymbol{L} = \dot{\boldsymbol{a}}_i\otimes\boldsymbol{a}^i, \quad \dot{\boldsymbol{a}}_i = \boldsymbol{L}\boldsymbol{a}_i, \\ &\bar{\boldsymbol{K}} = -\boldsymbol{a}_{3,\alpha}\otimes\boldsymbol{a}^\alpha = \bar{b}_{\alpha\beta}\boldsymbol{a}^\alpha\otimes\boldsymbol{a}^\beta, \end{aligned}\right\} \tag{8.1}$$

where $\bar{\boldsymbol{K}}$ is the curvature tensor with components $\bar{b}_{\alpha\beta}$. From (8.1) we write

$$\left.\begin{aligned} &\boldsymbol{L} = \boldsymbol{D}+\boldsymbol{W}, \quad \boldsymbol{D} = \boldsymbol{D}^{\mathrm{T}} = d^*_{ij}\boldsymbol{a}^i\otimes\boldsymbol{a}^j, \quad d^*_{i3} = 0, \quad 2d^*_{\alpha\beta} = \dot{a}_{\alpha\beta}, \\ &-\boldsymbol{W}^{\mathrm{T}} = \boldsymbol{W} = w^*_{ij}\boldsymbol{a}^i\otimes\boldsymbol{a}^j, \quad \boldsymbol{W}\boldsymbol{u} = \boldsymbol{\omega}\times\boldsymbol{u}, \quad \dot{\boldsymbol{a}}_3 = \boldsymbol{L}\boldsymbol{a}_3 = \boldsymbol{W}\boldsymbol{a}_3, \\ &w^*_{\alpha 3} = -w^*_{3\alpha} = -(v_{3,\alpha}+\bar{b}^\beta_\alpha v_\beta), \quad 2w^*_{\alpha\beta} = v_{\alpha|\beta}-v_{\beta|\alpha}, \quad w^*_{33} = 0, \quad \boldsymbol{v} = v_i\boldsymbol{a}^i, \end{aligned}\right\} \tag{8.2}$$

where (8.2) holds for every vector $\boldsymbol{u}$ and $\boldsymbol{\omega}$ is an axial vector.

The conservation laws for mass, momentum and moment of momentum are now given by (see also Naghdi 1972, §10):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\rho\,\mathrm{d}\sigma = 0, \tag{8.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\rho\boldsymbol{v}\,\mathrm{d}\sigma = \int_{\mathscr{P}}\rho(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})\,\mathrm{d}\sigma+\int_{\partial\mathscr{P}}\boldsymbol{n}\,\mathrm{d}s, \tag{8.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}(\boldsymbol{r}\times\boldsymbol{v}+y^{11}\bar{\boldsymbol{\omega}})\,\rho\,\mathrm{d}\sigma = \int_{\mathscr{P}}\{\boldsymbol{r}\times(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})+\bar{\boldsymbol{l}}+\bar{\boldsymbol{l}}_{\mathrm{e}}+\boldsymbol{c}_{\mathrm{e}}\}\rho\,\mathrm{d}\sigma+\int_{\partial\mathscr{P}}(\boldsymbol{r}\times\boldsymbol{n}+\hat{\boldsymbol{m}})\,\mathrm{d}s, \tag{8.5}$$

where $\hat{\boldsymbol{m}}$ is the edge couple and

$$\left.\begin{aligned}\hat{\boldsymbol{m}} = \boldsymbol{a}_3\times\boldsymbol{m},\quad \bar{\boldsymbol{\omega}} = \boldsymbol{a}_3\times(\boldsymbol{\omega}\times\boldsymbol{a}_3) = \omega^{\alpha}\boldsymbol{a}_{\alpha},\quad \boldsymbol{\omega} = \omega^{i}\boldsymbol{a}_{i},\\ \bar{\boldsymbol{l}}+\bar{\boldsymbol{l}}_{\mathrm{e}} = \boldsymbol{a}_3\times(\boldsymbol{l}+\boldsymbol{l}_{\mathrm{e}}),\quad \boldsymbol{c}_{\mathrm{e}} = c^{\alpha}\boldsymbol{a}_{\alpha}.\end{aligned}\right\} \tag{8.6}$$

Field equations that correspond to (8.3) to (8.5) are

$$\left.\begin{aligned}\dot{\rho}+\rho\,\mathrm{div}_s\,\boldsymbol{v} = 0 \quad\text{or}\quad \rho a^{\frac12} = \rho_{\mathrm{R}}A^{\frac12},\\ \rho\dot{\boldsymbol{v}} = \rho(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})+\mathrm{div}_s\,\boldsymbol{N},\\ \rho y^{11}\dot{\bar{\boldsymbol{\omega}}} = \rho(\bar{\boldsymbol{l}}+\bar{\boldsymbol{l}}_{\mathrm{e}})+\rho\boldsymbol{c}_{\mathrm{e}}+\boldsymbol{a}_{\alpha}\times\boldsymbol{N}^{\alpha}+\mathrm{div}_s\,\hat{\boldsymbol{M}},\end{aligned}\right\} \tag{8.7}$$

where

$$\left.\begin{aligned}\hat{\boldsymbol{m}} = \hat{\boldsymbol{M}}\boldsymbol{\nu} = \hat{\boldsymbol{M}}^{\alpha}\nu_{\alpha},\quad \hat{\boldsymbol{M}} = \hat{\boldsymbol{M}}^{\alpha}\otimes\boldsymbol{a}_{\alpha},\quad \hat{\boldsymbol{M}}^{\alpha} = \boldsymbol{a}_3\times\boldsymbol{M}^{\alpha} = M^{\beta\alpha}\boldsymbol{a}_3\times\boldsymbol{a}_{\beta},\\ \boldsymbol{n} = \boldsymbol{N}\boldsymbol{\nu} = \boldsymbol{N}^{\alpha}\nu_{\alpha},\quad \boldsymbol{N} = \boldsymbol{N}^{\alpha}\otimes\boldsymbol{a}_{\alpha},\quad \boldsymbol{N}^{\alpha} = N^{i\alpha}\boldsymbol{a}_{i}.\end{aligned}\right\} \tag{8.8}$$

The entropy balance equations are the same as (2.23) and (2.24) but the energy equation is now

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}(\epsilon+\tfrac12\boldsymbol{v}\cdot\boldsymbol{v}+\tfrac12y^{11}\bar{\boldsymbol{\omega}}\cdot\bar{\boldsymbol{\omega}})\rho\,\mathrm{d}\sigma = \int_{\mathscr{P}}\Big\{r+\sum_{N=1}^{K}r_N+(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})\cdot\boldsymbol{v}+(\bar{\boldsymbol{l}}+\bar{\boldsymbol{l}}_{\mathrm{e}})\cdot\boldsymbol{\omega}+\overline{w}\Big\}\rho\,\mathrm{d}\sigma\\ +\int_{\partial\mathscr{P}}\Big(\boldsymbol{n}\cdot\boldsymbol{v}+\hat{\boldsymbol{m}}\cdot\boldsymbol{\omega}-h-\sum_{N=1}^{K}h_N\Big)\,\mathrm{d}s.\end{aligned} \tag{8.9}$$

With the help of (8.7), the corresponding field equation reduces to

$$\left.\begin{aligned}\rho\Big(r+\sum_{N=1}^{K}r_N\Big)-\mathrm{div}_s\Big(\theta\boldsymbol{p}+\sum_{N=1}^{K}\theta_N\boldsymbol{p}_N\Big)-\rho\dot{\epsilon}+\rho\overline{w}-\rho\boldsymbol{c}_{\mathrm{e}}\cdot\boldsymbol{\omega}+P = 0,\\ P = \boldsymbol{N}^{\alpha}\cdot(\dot{\boldsymbol{a}}_{\alpha}-\boldsymbol{\omega}\times\boldsymbol{a}_{\alpha})+\hat{\boldsymbol{M}}^{\alpha}\cdot\boldsymbol{\omega}_{,\alpha} = \tfrac12(N^{\beta\alpha}+\bar{b}^{\beta}_{\lambda}M^{\alpha\lambda})\,\dot{a}_{\alpha\beta}-M^{\alpha\beta}\dot{\bar{b}}_{\alpha\beta},\end{aligned}\right\} \tag{8.10}$$

where

$$\left.\begin{aligned}\rho\overline{w}-\rho\boldsymbol{c}_{\mathrm{e}}\cdot\boldsymbol{\omega} = P_{\mathrm{e}}+\sum_{K=0}^{M}(\boldsymbol{E}_K\cdot\boldsymbol{J}_K+\boldsymbol{E}_K\cdot\dot{\boldsymbol{D}}_K+\boldsymbol{H}_K\cdot\dot{\boldsymbol{B}}_K)\,(A^{\frac12}/a^{\frac12}),\\ P_{\mathrm{e}} = \boldsymbol{N}^{\alpha}_{\mathrm{e}}\cdot(\dot{\boldsymbol{a}}_{\alpha}-\boldsymbol{\omega}\times\boldsymbol{a}_{\alpha})+\hat{\boldsymbol{M}}^{\alpha}_{\mathrm{e}}\cdot\boldsymbol{\omega}_{,\alpha} = \tfrac12(N^{\beta\alpha}_{\mathrm{e}}+\bar{b}^{\beta}_{\lambda}M^{\alpha\lambda}_{\mathrm{e}})\,\dot{a}_{\alpha\beta}-M^{\alpha\beta}_{\mathrm{e}}\dot{\bar{b}}_{\alpha\beta},\\ \rho\boldsymbol{c}_{\mathrm{e}} = \boldsymbol{a}_{\alpha}\times\boldsymbol{N}^{\alpha}_{\mathrm{e}}+\boldsymbol{a}_{3,\alpha}\times\boldsymbol{M}^{\alpha}_{\mathrm{e}}+\boldsymbol{a}_3\times\boldsymbol{k}_{\mathrm{e}}.\end{aligned}\right\} \tag{8.11}$$

With the help of (2.23) and (8.10) we obtain an energy identity of the form (2.25) but with P, P_{e} now given by (8.10) and (8.11).

Discussion of a magnetic thermoelastic shell now follows as in §4 except that the kinematic variables in (4.13) are replaced by

$$\boldsymbol{a}_{\alpha},\quad \boldsymbol{a}_{3,\alpha}. \tag{8.12}$$

With invariance under a constant rigid body rotation taken into account, we then have

$$\psi = \psi_5(a_{\alpha\beta},\bar{b}_{\alpha\beta},\theta,\theta_N,\boldsymbol{E}_M,\boldsymbol{H}_M,\boldsymbol{A}_{\alpha},\bar{B}_{\alpha\beta},\Theta;\theta^{\mu}). \tag{8.13}$$

Expressions for the entropies and the electromagnetic vectors are of the same form as those in (4.16), but $(4.16)_{1,2,3,4}$ are replaced by

$$\left.\begin{aligned}N^{\beta\alpha}+N^{\beta\alpha}_{\mathrm{e}}+\bar{b}^{\beta}_{\lambda}(M^{\alpha\lambda}+M^{\alpha\lambda}_{\mathrm{e}})+N^{\alpha\beta}+N^{\alpha\beta}_{\mathrm{e}}+\bar{b}^{\alpha}_{\lambda}(M^{\beta\lambda}+M^{\beta\lambda}_{\mathrm{e}}) = 2\rho\Big(\frac{\partial\psi_5}{\partial a_{\alpha\beta}}+\frac{\partial\psi_5}{\partial a_{\beta\alpha}}\Big),\\ M^{\alpha\beta}+M^{\alpha\beta}_{\mathrm{e}}+M^{\beta\alpha}+M^{\beta\alpha}_{\mathrm{e}} = -\rho\Big(\frac{\partial\psi_5}{\partial\bar{b}_{\alpha\beta}}+\frac{\partial\psi_5}{\partial\bar{b}_{\beta\alpha}}\Big).\end{aligned}\right\} \tag{8.14}$$

In addition, taking the scalar product of each side of $(8.7)_3$ with $\boldsymbol{a}_3$ yields

$$N^{\beta\alpha}+N_{\mathrm{e}}^{\beta\alpha}+\bar{b}^{\beta}_{\lambda}(M^{\alpha\lambda}+M_{\mathrm{e}}^{\alpha\lambda}) = N^{\alpha\beta}+N_{\mathrm{e}}^{\alpha\beta}+\bar{b}^{\alpha}_{\lambda}(M^{\beta\lambda}+M_{\mathrm{e}}^{\beta\lambda}). \tag{8.15}$$

To complete the constitutive equations, we set

$$M^{\alpha\beta}+M_{\mathrm{e}}^{\alpha\beta} = M^{\beta\alpha}+M_{\mathrm{e}}^{\beta\alpha}. \tag{8.16}$$

In parallel with § 6 we limit further discussion to the linear theory of a plate of constant thickness, unstressed, at uniform temperature $\bar{\theta}$ and without electromagnetic fields in its reference configuration. The middle plane and its motion are specified by

$$\boldsymbol{R} = x_{\alpha}\boldsymbol{e}_{\alpha}, \quad \boldsymbol{r} = \boldsymbol{R}+\boldsymbol{u}, \quad \boldsymbol{u} = u_i\boldsymbol{e}_i, \quad \boldsymbol{v} = \dot{\boldsymbol{u}}, \quad \theta^{\alpha} = x_{\alpha}. \tag{8.17}$$

Linear kinematic measures are

$$e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha}), \quad \kappa_{\alpha\beta} = -u_{3,\alpha\beta}, \tag{8.18}$$

where $\kappa_{\alpha\beta}$ is the curvature of the deformed surface. All response functions are now referred to the reference body with

$$\left.\begin{aligned} \boldsymbol{n} = \boldsymbol{N}\boldsymbol{\nu} = \boldsymbol{N}_{\alpha}\nu_{\alpha}, \quad \boldsymbol{N} = \boldsymbol{N}_{\alpha}\otimes\boldsymbol{e}_{\alpha}, \quad \boldsymbol{N}_{\alpha} = N_{i\alpha}\boldsymbol{e}_i, \\ \hat{\boldsymbol{m}} = \hat{\boldsymbol{M}}\boldsymbol{\nu} = \hat{\boldsymbol{M}}_{\alpha}\nu_{\alpha}, \quad \hat{\boldsymbol{M}} = \hat{\boldsymbol{M}}_{\alpha}\otimes\boldsymbol{e}_{\alpha}, \quad \hat{\boldsymbol{M}}_{\alpha} = \boldsymbol{e}_3\times\boldsymbol{M}_{\alpha} = M_{\beta\alpha}\boldsymbol{e}_3\times\boldsymbol{e}_{\beta}. \end{aligned}\right\} \tag{8.19}$$

Component forms of the equations of motion (8.7) reduce to

$$\rho\ddot{u}_i = \rho f_i+N_{i\alpha,\alpha}, \quad N_{\alpha\beta} = N_{\beta\alpha}, \quad -\rho y^{11}\ddot{u}_{3,\alpha} = \rho l_{\alpha}+M_{\alpha\beta,\beta}-N_{3\alpha}, \quad M_{\alpha\beta} = M_{\beta\alpha}, \tag{8.20}$$

where ρ is reference density.

We now restrict attention to the geometric symmetry condition (5.3*a*) or (5.3*b*) and, instead of (6.16), for the Helmholtz free energy response functions write

$$\begin{aligned} \rho\psi = {} & \tfrac{1}{2}A_{\alpha\beta\lambda\mu}e_{\alpha\beta}e_{\lambda\mu}+\tfrac{1}{2}B_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu}-\tfrac{1}{2}P\theta^2-P_{\alpha\beta}e_{\alpha\beta}\theta-\tfrac{1}{2}Q\phi^2-Q_{\alpha\beta}\kappa_{\alpha\beta}\phi \\ & -e_{\alpha\beta}\sum_{M=0}^{L}(C^{M}_{\alpha\beta i}E_{Mi}+F^{M}_{\alpha\beta i}H_{Mi})-\kappa_{\alpha\beta}\sum_{M=0}^{L}(\bar{C}^{M}_{\alpha\beta i}E_{Mi}+\bar{F}^{M}_{\alpha\beta i}H_{Mi}) \\ & +\theta\sum_{M=0}^{L}(R^{M}_{i}E_{Mi}+S^{M}_{i}H_{Mi})+\phi\sum_{M=0}^{L}(\bar{R}^{M}_{i}E_{Mi}+\bar{S}^{M}_{i}H_{Mi}) \\ & -\sum_{M=0}^{L}\sum_{N=0}^{L}(\tfrac{1}{2}L^{MN}_{ij}E_{Mi}E_{Nj}+M^{MN}_{ij}E_{Mi}H_{Nj}+\tfrac{1}{2}N^{MN}_{ij}H_{Mi}H_{Nj}). \end{aligned} \tag{8.21}$$

With the help of (8.21) and the linearized forms of (8.14), we obtain the constitutive relations

$$\left.\begin{aligned} N_{\alpha\beta} = N_{\beta\alpha} &= A_{\alpha\beta\lambda\mu}e_{\lambda\mu}-P_{\alpha\beta}\theta-\sum_{M=0}^{L}(C^{M}_{\alpha\beta i}E_{Mi}+F^{M}_{\alpha\beta i}H_{Mi}), \\ M_{\alpha\beta} = M_{\beta\alpha} &= B_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}-Q_{\alpha\beta}\phi-\sum_{M=0}^{L}(\bar{C}^{M}_{\alpha\beta i}E_{Mi}+\bar{F}^{M}_{\alpha\beta i}H_{Mi}), \\ \rho\eta &= P_{\alpha\beta}e_{\alpha\beta}+P\theta-\sum_{M=0}^{L}(R^{M}_{i}E_{Mi}+S^{M}_{i}H_{Mi}), \\ \rho\eta_1 &= Q_{\alpha\beta}\kappa_{\alpha\beta}+Q\phi-\sum_{M=0}^{L}(\bar{R}^{M}_{i}E_{Mi}+\bar{S}^{M}_{i}H_{Mi}), \\ \bar{D}_{Mi} &= C^{M}_{\alpha\beta i}e_{\alpha\beta}+\bar{C}^{M}_{\alpha\beta i}\kappa_{\alpha\beta}-R^{M}_{i}\theta-\bar{R}^{M}_{i}\phi+\sum_{N=0}^{L}(L^{MN}_{ij}E_{Nj}+M^{MN}_{ij}H_{Nj}), \\ B_{Mi} &= F^{M}_{\alpha\beta i}e_{\alpha\beta}+\bar{F}^{M}_{\alpha\beta i}\kappa_{\alpha\beta}-S^{M}_{i}\theta-\bar{S}^{M}_{i}\phi+\sum_{N=0}^{L}(M^{NM}_{ji}E_{Nj}+N^{MN}_{ij}H_{Nj}), \end{aligned}\right\} \tag{8.22}$$

with partially inverse relations

$$G = G(N_{\alpha\beta}, M_{\alpha\beta}, \theta, \phi, E_{Mi}, H_{Mi}),$$

$$e_{\alpha\beta} = -\tfrac{1}{2}\rho\left(\frac{\partial G}{\partial N_{\alpha\beta}} + \frac{\partial G}{\partial N_{\beta\alpha}}\right), \quad \kappa_{\alpha\beta} = -\tfrac{1}{2}\rho\left(\frac{\partial G}{\partial M_{\alpha\beta}} + \frac{\partial G}{\partial M_{\beta\alpha}}\right),$$

$$\eta = -\frac{\partial G}{\partial \theta}, \quad \eta_1 = -\frac{\partial G}{\partial \phi}, \quad \bar{D}_{Mi} = -\rho\frac{\partial G}{\partial E_{Mi}}, \quad B_{Mi} = -\rho\frac{\partial G}{\partial H_{Mi}},$$

where

$$\begin{aligned}\rho G = &-\tfrac{1}{2}A^*_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu} - \tfrac{1}{2}B^*_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu} - \tfrac{1}{2}P^*\theta^2 - P^*_{\alpha\beta}N_{\alpha\beta}\theta - \tfrac{1}{2}Q^*\phi^2 - Q^*_{\alpha\beta}M_{\alpha\beta}\\ &+ N_{\alpha\beta}\sum_{M=0}^{L}(C^{*M}_{\alpha\beta i}E_{Mi} + F^{*M}_{\alpha\beta i}H_{Mi}) + M_{\alpha\beta}\sum_{M=0}^{L}(\bar{C}^{*M}_{\alpha\beta i}E_{Mi} + \bar{F}^{*M}_{\alpha\beta i}H_{Mi})\\ &+ \theta\sum_{M=0}^{L}(R^{*M}_{i}E_{Mi} + S^{*M}_{i}H_{Mi}) + \phi\sum_{M=0}^{L}(\bar{R}^{*M}_{i}E_{Mi} + \bar{S}^{*M}_{i}H_{Mi})\\ &+ \sum_{M=0}^{L}\sum_{N=0}^{L}(\tfrac{1}{2}L^{*MN}_{ij}E_{Mi}E_{Nj} + M^{*MN}_{ij}E_{Mi}H_{Nj} + \tfrac{1}{2}N^{*MN}_{ij}H_{Mi}H_{Nj})\end{aligned} \tag{8.23}$$

and

$$\left.\begin{aligned} A_{\alpha\beta\lambda\mu}A^*_{\lambda\mu\rho\nu} = B_{\alpha\beta\lambda\mu}B^*_{\lambda\mu\rho\nu} &= \tfrac{1}{2}(\delta_{\alpha\rho}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\rho}),\\ A_{\alpha\beta\lambda\mu}P^*_{\lambda\mu} - P_{\alpha\beta} = 0, &\quad P - P^* + P_{\alpha\beta}P^*_{\alpha\beta} = 0,\\ B_{\alpha\beta\lambda\mu}Q^*_{\lambda\mu} - Q_{\alpha\beta} = 0, &\quad Q - Q^* + Q_{\alpha\beta}Q^*_{\alpha\beta} = 0,\\ A_{\alpha\beta\lambda\mu}C^{*M}_{\alpha\beta i} + C^{M}_{\alpha\beta i} = 0, &\quad A_{\alpha\beta\lambda\mu}F^{*M}_{\lambda\mu i} + F^{M}_{\alpha\beta i} = 0,\\ R^{*M}_{i} = R^{M}_{i} - P^*_{\alpha\beta}C^{M}_{\alpha\beta i}, &\quad S^{*M}_{i} = S^{M}_{i} - P^*_{\alpha\beta}F^{M}_{\alpha\beta i},\\ B_{\alpha\beta\lambda\mu}\bar{C}^{*M}_{\lambda\mu i} + \bar{C}^{M}_{\alpha\beta i} = 0, &\quad B_{\alpha\beta\lambda\mu}\bar{F}^{*M}_{\lambda\mu i} + \bar{F}^{M}_{\alpha\beta i} = 0,\\ \bar{R}^{*M}_{i} = \bar{R}^{M}_{i} - Q^*_{\alpha\beta}\bar{C}^{M}_{\alpha\beta i}, &\quad \bar{S}^{*M}_{i} = \bar{S}^{M}_{i} - Q^*_{\alpha\beta}\bar{F}^{M}_{\alpha\beta i},\\ L^{*MN}_{ij} + L^{MN}_{ij} &= C^{*M}_{\alpha\beta i}C^{N}_{\alpha\beta j} + \bar{C}^{*M}_{\alpha\beta i}\bar{C}^{N}_{\alpha\beta j},\\ M^{*MN}_{ij} + M^{MN}_{ij} &= C^{*M}_{\alpha\beta i}F^{N}_{\alpha\beta j} + \bar{C}^{*M}_{\alpha\beta i}\bar{F}^{N}_{\alpha\beta j},\\ N^{*MN}_{ij} + N^{MN}_{ij} &= F^{*M}_{\alpha\beta i}F^{N}_{\alpha\beta j} + \bar{F}^{*M}_{\alpha\beta i}\bar{F}^{N}_{\alpha\beta j}.\end{aligned}\right\} \tag{8.24}$$

There are also similar formulae in which starred and unstarred symbols are interchanged.

Values of the coefficients in (8.23) are the same as those listed in (6.25)–(6.27) for case (*a*) and in (6.25) and (6.29) for case (*b*), for the relevant coefficients.

The constitutive equations for p_α, $p_{1\alpha}$, ξ, ξ_1, J_{Mi} are still given by (6.28) for case (*a*).

9. Membrane theory

For some purposes, the mechanical properties of a shell can be examined by using only the membrane theory, but it is desirable to keep enough generality in the thermal and electromagnetic part of the theory to allow for effects of these, both along and across the material surface of the membrane. The nonlinear membrane theory can be obtained as a special case of the theory in §§ 2–4 merely by suppressing the director and the associated response functions, but it is convenient to list here the main results for a magnetic, polarized thermoelastic membrane.

From (2.14), (2.16), (2.31) and (2.33), the spatial and material forms of the equations of motion are given respectively by

$$\rho\dot{\boldsymbol{v}} = \rho(\boldsymbol{f} + \boldsymbol{f}_{\mathrm{e}}) + \mathrm{div}_s\,\boldsymbol{N}, \quad \rho\boldsymbol{\Gamma}_{\mathrm{e}} + \boldsymbol{N} - \boldsymbol{N}^{\mathrm{T}} = \boldsymbol{0} \quad \text{or} \quad \rho\boldsymbol{c}_{\mathrm{e}} + \boldsymbol{a}_\alpha \times \boldsymbol{N}^\alpha = \boldsymbol{0}, \tag{9.1}$$

and

$$\rho_{\mathrm{R}}\,\dot{\boldsymbol{v}} = \rho_{\mathrm{R}}(\boldsymbol{f}+\boldsymbol{f}_{\mathrm{e}})+\mathrm{Div}_{s\,\mathrm{R}}\boldsymbol{N}, \quad \rho_{\mathrm{R}}\boldsymbol{\Gamma}_{\mathrm{e}}+{}_{\mathrm{R}}\boldsymbol{N}\boldsymbol{F}^{\mathrm{T}}-\boldsymbol{F}_{\mathrm{R}}\boldsymbol{N}^{\mathrm{T}} = \boldsymbol{0} \quad \text{or} \quad \rho_{\mathrm{R}}\,\boldsymbol{c}_{\mathrm{e}}+\boldsymbol{a}_{\alpha}\times{}_{\mathrm{R}}\boldsymbol{N}^{\alpha} = \boldsymbol{0}, \tag{9.2}$$

where

$$\left.\begin{aligned} \boldsymbol{n} = \boldsymbol{N}\boldsymbol{v} = \boldsymbol{N}^{\alpha}\nu_{\alpha}, \quad \boldsymbol{N} = \boldsymbol{N}^{\alpha}\otimes\boldsymbol{a}_{\alpha}, \quad \boldsymbol{N}^{\alpha} = N^{i\alpha}\boldsymbol{a}_{i}, \\ {}_{\mathrm{R}}\boldsymbol{n} = {}_{\mathrm{R}}\boldsymbol{N}_{\mathrm{R}}\boldsymbol{v} = {}_{\mathrm{R}}\boldsymbol{N}^{\alpha}{}_{\mathrm{R}}\nu_{\alpha}, \quad {}_{\mathrm{R}}\boldsymbol{N} = {}_{\mathrm{R}}\boldsymbol{N}^{\alpha}\otimes\boldsymbol{A}_{\alpha}, \quad {}_{\mathrm{R}}\boldsymbol{N}^{\alpha} = {}_{\mathrm{R}}N^{i\alpha}\boldsymbol{A}_{i}. \end{aligned}\right\} \tag{9.3}$$

The entropy balance equations are still given by (2.23) or by their corresponding material forms. The electromagnetic field equations are either (3.7)–(3.10) or, in material form, (3.17)–(3.20).

From (4.12) and (4.15), the constitutive equations in terms of the Helmholtz free energy function reduce to

$$\left.\begin{aligned} \psi = \psi_3(\boldsymbol{F},\theta,\theta_N,\boldsymbol{E}_M,\boldsymbol{H}_M), \\ \eta = -\frac{\partial\psi_3}{\partial\theta}, \quad \eta_N = -\frac{\partial\psi_3}{\partial\theta_N}, \quad {}_{\mathrm{R}}\boldsymbol{N}+{}_{\mathrm{R}}\boldsymbol{N}_{\mathrm{e}} = \rho_{\mathrm{R}}\frac{\partial\psi_3}{\partial\boldsymbol{F}}, \\ \bar{\boldsymbol{D}}_M = -\rho_{\mathrm{R}}\frac{\partial\psi_3}{\partial\boldsymbol{E}_M}, \quad \boldsymbol{B}_M = -\rho_{\mathrm{R}}\frac{\partial\psi_3}{\partial\boldsymbol{H}_M}, \quad \frac{\partial\psi_3}{\partial\boldsymbol{F}}\boldsymbol{A}_3 = \boldsymbol{0}, \quad \boldsymbol{F} = \boldsymbol{a}_i\otimes\boldsymbol{A}^i, \end{aligned}\right\} \tag{9.4}$$

or

$$\left.\begin{aligned} \psi = \psi_5(a_{\alpha\beta},\theta,\theta_N,E_{Mi},H_{Mi}), \\ N^{\beta\alpha}+N_{\mathrm{e}}^{\beta\alpha} = \rho\left(\frac{\partial\psi_5}{\partial a_{\alpha\beta}}+\frac{\partial\psi_5}{\partial a_{\beta\alpha}}\right) = (A^{\frac{1}{2}}/a^{\frac{1}{2}})\,({}_{\mathrm{R}}N^{k\alpha}+{}_{\mathrm{R}}N_{\mathrm{e}}^{k\alpha})\,\boldsymbol{A}_k\cdot\boldsymbol{a}^{\beta}, \\ (a^{\frac{1}{2}}/A^{\frac{1}{2}})\,(N^{3\alpha}+N_{\mathrm{e}}^{3\alpha}) = ({}_{\mathrm{R}}N^{k\alpha}+{}_{\mathrm{R}}N_{\mathrm{e}}^{k\alpha})\,\boldsymbol{A}_k\cdot\boldsymbol{a}^3 = 0, \\ \eta = -\frac{\partial\psi_5}{\partial\theta}, \quad \eta_N = -\frac{\partial\psi_5}{\partial\theta_N}, \quad \bar{D}_M^i = -\rho_{\mathrm{R}}\frac{\partial\psi_5}{\partial E_{Mi}}, \quad B_M^i = -\rho_{\mathrm{R}}\frac{\partial\psi_5}{\partial H_{Mi}}. \end{aligned}\right\} \tag{9.5}$$

Linearized membrane theory follows in a usual way. Let $\boldsymbol{u}$ denote the displacement vector of the material surface $\mathscr{S}$ of the membrane. Then,

$$\boldsymbol{u} = u_i\boldsymbol{A}^i = u^i\boldsymbol{A}_i, \quad e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha})-\bar{B}_{\alpha\beta}u_3, \tag{9.6}$$

where $e_{\alpha\beta}$ is the surface strain, a vertical line stands for covariant differentiation with respect to the metric tensor of the surface $\mathscr{S}$ in the reference configuration, and $\bar{B}_{\alpha\beta}$ is the curvature tensor of $\mathscr{S}$ in the reference configuration. As in § 6, we omit all suffices R and refer all quantities to the reference surface, which we assume to be unstressed, at constant temperature $\bar{\theta}$ and without electromagnetic fields. Then, the component form of the equations of motion (9.1) when $\boldsymbol{N}^{\alpha}$ are referred to the basis $\boldsymbol{A}_i$ in the reference configuration is:

$$N^{\alpha\beta}{}_{|\beta}+\rho f^{\alpha} = \rho\ddot{u}^{\alpha}, \quad N^{\alpha\beta} = N^{\beta\alpha}, \quad \bar{B}_{\alpha\beta}N^{\alpha\beta}+\rho f^3 = \rho\ddot{u}^3, \quad N^{3\alpha} = 0, \quad \boldsymbol{f} = f^i\boldsymbol{A}_i = f_i\boldsymbol{A}^i, \tag{9.7}$$

and the entropy balance equations are

$$\left.\begin{aligned} \rho\dot{\eta} = \rho(s+\xi)-p^{\alpha}{}_{|\alpha}, \quad \boldsymbol{p} = p^{\alpha}\boldsymbol{A}_{\alpha} = p_{\alpha}\boldsymbol{A}^{\alpha}, \quad k = p^{\alpha}\nu_{\alpha}, \\ \rho\dot{\eta}_N = \rho(s_N+\xi_N)-p^{\alpha}_{N|\alpha}, \quad \boldsymbol{p}_N = p^{\alpha}_N\boldsymbol{A}_{\alpha} = p_{N\alpha}\boldsymbol{A}^{\alpha}, \quad k_N = p^{\alpha}_N\nu_{\alpha}, \end{aligned}\right\} \tag{9.8}$$

where ρ is reference density and $\boldsymbol{v} = \nu_{\alpha}\boldsymbol{A}^{\alpha} = \nu^{\alpha}\boldsymbol{A}_{\alpha}$ is the outward unit normal to any closed curve in the reference surface. If the reference surface is one of constant thickness h, we choose $\boldsymbol{D}_1 = \boldsymbol{A}_3$, $\boldsymbol{D}_N = \boldsymbol{0}$ ($N \geqslant 2$) in (B 1) so that the electromagnetic field equations in (3.17)–(3.20) are

$$\left.\begin{aligned} B^{\alpha}_{M|\alpha} = \sum_{K=0}^{M}\chi^K_M B^3_K-[\chi_M(z)\,B^3]^{\frac{1}{2}h}_{-\frac{1}{2}h}, \quad \bar{D}^{\alpha}_{M|\alpha} = \sum_{K=0}^{M}\psi^K_M\bar{D}^3_K-[\psi_M(z)\,\bar{D}^3]^{\frac{1}{2}h}_{-\frac{1}{2}h}+E_M, \\ \dot{B}^3_M = -\epsilon^{\alpha\beta}E_{M\beta,\alpha}, \quad \dot{B}^{\alpha}_M = -\epsilon^{\alpha\beta}\left\{E_{M3,\beta}+\sum_{K=0}^{M}\chi^K_M E_{K\beta}-[\chi_M(z)\,E_{\beta}]^{\frac{1}{2}h}_{-\frac{1}{2}h}\right\}, \\ -\dot{\bar{D}}^3_M = J^3_M-\epsilon^{\alpha\beta}H_{M\beta,\alpha}, \quad -\dot{\bar{D}}^{\alpha}_M = J^{\alpha}_M-\epsilon^{\alpha\beta}\left\{H_{M3,\beta}+\sum_{K=0}^{M}\psi^K_M H_{K\beta}-[\psi_M(z)\,H_{\beta}]^{\frac{1}{2}h}_{-\frac{1}{2}h}\right\}, \end{aligned}\right\} \tag{9.9}$$

where $\epsilon^{\alpha\beta}$ is now the alternating tensor of the reference surface. In writing the surface terms $[\ ]_{-\frac{1}{2}h}^{\frac{1}{2}h}$ on the right-hand sides of (9.9) we have used the approximation $h/R \ll 1$, where h denotes the thickness of the membrane and R is smallest radius of the curvature at any point of the material surface $\mathscr{S}$.

Restricting attention to the conditions (5.3*a*) or (5.3*b*), which arise from geometrical symmetry conditions in the shell-like body, and assuming a quadratic expression for the Helmholtz energy response function, we have

$$\begin{aligned}\rho\psi = {}& \tfrac{1}{2}A^{\alpha\beta\lambda\mu}e_{\alpha\beta}e_{\lambda\mu} - \tfrac{1}{2}P\theta^2 - P^{\alpha\beta}e_{\alpha\beta}\theta - \tfrac{1}{2}Q\phi^2 - e_{\alpha\beta}\sum_{M=0}^{L}(C^{M\alpha\beta i}E_{Mi} + F^{M\alpha\beta i}H_{Mi}) \\ & + \theta\sum_{M=0}^{L}(R^{Mi}E_{Mi} + S^{Mi}H_{Mi}) + \phi\sum_{M=0}^{L}(\bar{R}^{Mi}E_{Mi} + \bar{S}^{Mi}H_{Mi}) \\ & - \sum_{M=0}^{L}\sum_{N=0}^{L}(\tfrac{1}{2}L^{MNij}E_{Mi}E_{Nj} + M^{MNij}E_{Mi}H_{Nj} + N^{MNij}H_{Mi}H_{Nj}). \end{aligned} \tag{9.10}$$

Latin lower case indices are raised or lowered with the help of the metric tensors A_{ij}, A^{ij}, where

$$A_{\alpha 3} = 0, \quad A^{\alpha 3} = 0, \quad A_{33} = A^{33} = 1. \tag{9.11}$$

From (9.5) and (9.10) we have

$$\left.\begin{aligned} N^{\alpha\beta} = N^{\beta\alpha} &= A^{\alpha\beta\lambda\mu}e_{\lambda\mu} - P^{\alpha\beta}\theta - \sum_{M=0}^{L}(C^{M\alpha\beta i}E_{Mi} + F^{M\alpha\beta i}H_{Mi}), \\ \rho\eta &= P^{\alpha\beta}e_{\alpha\beta} + P\theta - \sum_{M=0}^{L}(R^{Mi}E_{Mi} + S^{Mi}H_{Mi}), \\ \rho\eta_1 &= Q\phi - \sum_{M=0}^{L}(\bar{R}^{Mi}E_{Mi} + \bar{S}^{Mi}H_{Mi}), \\ \bar{D}^i_M &= C^{M\alpha\beta i}e_{\alpha\beta} - R^{Mi}\theta - \bar{R}^{Mi}\phi + \sum_{N=0}^{L}(L^{MNij}E_{Nj} + M^{MNij}H_{Nj}), \\ B^i_M &= F^{M\alpha\beta i}e_{\alpha\beta} - S^{Mi}\theta - \bar{S}^{Mi}\phi + \sum_{N=0}^{L}(M^{NMji}E_{Nj} + N^{MNij}H_{Nj}). \end{aligned}\right\} \tag{9.12}$$

Partially inverse relations with respect to $e_{\alpha\beta}$, $N^{\alpha\beta}$ are:

$$\left.\begin{aligned} & G = G(N^{\alpha\beta}, \theta, \phi, E_{Mi}, H_{Mi}), \\ & e_{\alpha\beta} = -\tfrac{1}{2}\rho\left(\frac{\partial G}{\partial N^{\alpha\beta}} + \frac{\partial G}{\partial N^{\beta\alpha}}\right), \quad \eta = -\frac{\partial G}{\partial\theta}, \quad \eta_1 = -\frac{\partial G}{\partial\theta}, \\ & \bar{D}^i_M = -\rho\frac{\partial G}{\partial E_{Mi}}, \quad B^i_M = -\rho\frac{\partial G}{\partial H_{Mi}}, \end{aligned}\right\} \tag{9.13}$$

where

$$\begin{aligned}\rho G = {}& -\tfrac{1}{2}A^*_{\alpha\beta\lambda\mu}N^{\alpha\beta}N^{\lambda\mu} - \tfrac{1}{2}P^*\theta^2 - P^*_{\alpha\beta}N^{\alpha\beta}\theta - \tfrac{1}{2}Q^*\phi^2 + N^{\alpha\beta}\sum_{M=0}^{L}(C^{*Mi}_{\alpha\beta}E_{Mi} + F^{*Mi}_{\alpha\beta}H_{Mi}) \\ & + \theta\sum_{M=0}^{L}(R^{*Mi}E_{Mi} + S^{*Mi}H_{Mi}) + \phi\sum_{M=0}^{L}(\bar{R}^{*Mi}E_{Mi} + \bar{S}^{*Mi}H_{Mi}) \\ & + \sum_{M=0}^{L}\sum_{N=0}^{L}(\tfrac{1}{2}L^{*MNij}E_{Mi}E_{Nj} + M^{MNij}E_{Mi}H_{Nj} + N^{MNij}H_{Mi}H_{Nj}) \end{aligned} \tag{9.14}$$

and
$$\left.\begin{aligned} &A^{\alpha\beta\lambda\mu}A^{*}_{\lambda\mu\rho\nu} = \tfrac{1}{2}(\delta^{\alpha}_{\rho}\,\delta^{\beta}_{\nu} + \delta^{\alpha}_{\nu}\,\delta^{\beta}_{\rho}),\\ &A^{\alpha\beta\lambda\mu}P^{*}_{\lambda\mu} - P^{\alpha\beta} = 0, \quad P - P^{*} + P^{\alpha\beta}P^{*}_{\alpha\beta} = 0, \quad Q = Q^{*},\\ &A^{\alpha\beta\lambda\mu}C^{*Mi}_{\lambda\mu} + C^{M\alpha\beta i} = 0, \quad A^{\alpha\beta\lambda\mu}F^{*Mi}_{\lambda\mu} + F^{M\alpha\beta i} = 0,\\ &R^{*Mi} = R^{Mi} - P^{*}_{\alpha\beta}C^{M\alpha\beta i}, \quad S^{*Mi} = S^{Mi} - P^{*}_{\alpha\beta}F^{M\alpha\beta i},\\ &\bar{R}^{*Mi} = \bar{R}^{Mi}, \quad \bar{S}^{*Mi} = \bar{S}^{Mi},\\ &L^{*MNij} + L^{MNij} = C^{*Mi}_{\alpha\beta}C^{N\alpha\beta j},\\ &M^{*MNij} + M^{MNij} = C^{*Mi}_{\alpha\beta}F^{N\alpha\beta j},\\ &N^{*MNij} + N^{MNij} = F^{*Mi}_{\alpha\beta}F^{N\alpha\beta j}. \end{aligned}\right\} \tag{9.15}$$

Values of the coefficients in (9.15) are the same as those listed in (6.25)–(6.27) for case (*a*) and in (6.25) to (6.29) for case (*b*), provided that in these formulae and in the relations (D 5) $C^{*M}_{\alpha\beta i}$, $F^{*M}_{\alpha\beta i}$, R^{*M}_{i}, S^{*M}_{i}, $\bar{R}^{*M}_{i}$, $\bar{S}^{*M}_{i}$, L^{*MN}_{ij}, M^{*MN}_{ij}, $k^{*}_{\alpha\beta i}$, $l^{*}_{\alpha\beta i}$, f^{*}_{i}, g^{*}_{i}, f^{*}_{ij}, g^{*}_{ij} are replaced, respectively, by $C^{*Mi}_{\alpha\beta}$, $F^{*Mi}_{\alpha\beta}$, R^{*Mi}, S^{*Mi}, $\bar{R}^{*Mi}$, $\bar{S}^{*Mi}$, L^{*MNij}, M^{*MNij}, N^{*MNij}, ${k^{*}}_{\alpha\beta}{}^{i}$, ${l^{*}}_{\alpha\beta}{}^{i}$, f^{*i}, g^{*i}, f^{*ij}, g^{*ij}. Also, the terms c_{ijrs}, c_{ij}, ... in (D 5) are replaced by c^{ijrs}, c^{ij}, ... in (D 8).

The corresponding constitutive equations in case (*a*) for p^{α}, p^{α}_{1}, ξ, ξ_1, J^{i}_{0}, J^{i}_{1}, J^{i}_{M} from (D 9) are:

$$\left.\begin{aligned} p^{\alpha} &= -hk^{\alpha\beta}\theta_{,\beta} - hk^{\alpha3}\phi - \bar{a}^{\alpha i}h^{\frac{1}{2}}E_{0i},\\ p^{\alpha}_{1} &= -\tfrac{1}{12}h^{3}k^{\alpha\beta}\phi_{,\beta} - (h^{\frac{1}{2}}/c^{0}_{1})\,\bar{a}^{\alpha i}E_{1i},\\ \xi = 0, \quad \rho\xi_{1} &= -hk^{3\alpha}\theta_{,\alpha} - hk^{33}\phi - h^{\frac{1}{2}}\bar{a}^{3i}E_{0i},\\ J^{i}_{0} &= h^{\frac{1}{2}}(l^{i\alpha}\theta_{,\alpha} + l^{i3}\phi) + b^{ij}E_{0j},\\ J^{i}_{1} &= (h^{\frac{1}{2}}/c^{0}_{1})\,l^{i\alpha}\phi_{,\alpha} + b^{ij}E_{1j},\\ J^{i}_{M} &= b^{ij}E_{Mj} \quad (M = 2, 3, \ldots, L). \end{aligned}\right\} \tag{9.16}$$

10. Thermomechanical and electromagnetic effects in a non-conducting plate

The various linear theories of plates discussed in §§ 6–8 include both thermal and electromagnetic effects, but differ from one another only to the extent in which the effect of transverse deformations, i.e. (i) the transverse shear deformation and (ii) the transverse normal strain (through appropriate kinematical variables) are accounted for. The effects of both (i) and (ii) are included in the theory of § 6; both effects are suppressed in the restricted theory of § 8; and only the effect of (i) is retained in the partially restricted theory of § 7. As one example we consider a non-conducting plate in free space and in the absence of the effects of body force and applied traction on the major surfaces $z = \pm\frac{1}{2}h$, which are also heat insulated. We use here the theory of § 7, but similar analyses can be made from the theories of §§ 6 and 8. For the electromagnetic part of the theory, we select the representation corresponding to case (*a*) in (6.19) and restrict the electromagnetic variables to $\boldsymbol{E}_0$, $\boldsymbol{E}_1$, $\boldsymbol{H}_0$, $\boldsymbol{H}_1$, $\bar{\boldsymbol{D}}_0$, $\bar{\boldsymbol{D}}_1$, $\boldsymbol{B}_0$, $\boldsymbol{B}_1$. It follows that for the example under discussion

$$s = 0, \quad s_1 = 0, \quad f_i = 0, \quad l_\beta = 0, \quad \boldsymbol{J}_M = 0, \quad E^{*}_{M} = 0. \tag{10.1}$$

With the help of (6.28), the equations of motion (7.2) and entropy balance equations (6.8) for case (*a*) reduce to

$$\rho\ddot{u}_i = N_{i\alpha,\alpha}, \quad \rho y^{11}\ddot{\delta}_\alpha = -k_\alpha + M_{\alpha\beta,\beta}, \quad k_\alpha = N_{3\alpha}, \quad \rho = \rho^{*}h, \quad y^{11} = \tfrac{1}{12}h^2, \tag{10.2}$$

$$\left.\begin{aligned} \rho\dot{\eta} &= hk_{\alpha\beta}\theta_{,\alpha\beta} + hk_{\alpha3}\phi_{,\alpha} + h^{\frac{1}{2}}\bar{a}_{\alpha i}E_{0i,\alpha},\\ \rho\dot{\eta}_1 &= -hk_{3\alpha}\theta_{,\alpha} - hk_{33}\phi - h^{\frac{1}{2}}\bar{a}_{3i}E_{0i} + \tfrac{1}{12}h^3k_{\alpha\beta}\phi_{,\alpha\beta} + (h^{\frac{1}{2}}/c^{0}_{1})\,\bar{a}_{\alpha i}E_{1i,\alpha}, \quad c^{0}_{1} = 2.3^{\frac{1}{2}}h^{-1}. \end{aligned}\right\} \tag{10.3}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The relevant electromagnetic field equations in (6.11) are

$$\left.\begin{aligned} \dot{\bar{D}}_{03} &= \epsilon_{\alpha\beta} H_{0\beta,\alpha}, \quad \dot{\bar{D}}_{0\alpha} = \epsilon_{\alpha\beta} H_{03,\beta} - \epsilon_{\alpha\beta} \hat{H}_{0\beta}, \\ \dot{B}_{03} &= -\epsilon_{\alpha\beta} E_{0\beta,\alpha}, \quad \dot{B}_{0\alpha} = -\epsilon_{\alpha\beta} E_{03,\beta} + \epsilon_{\alpha\beta} \hat{E}_{0\beta}, \\ \bar{D}_{0\alpha,\alpha} &= -\hat{D}_{03}, \quad B_{0\alpha,\alpha} = -\hat{B}_{03}, \end{aligned}\right\} \tag{10.4}$$

$$\left.\begin{aligned} \dot{\bar{D}}_{13} &= \epsilon_{\alpha\beta} H_{1\beta,\alpha}, \quad \dot{\bar{D}}_{1\alpha} = \epsilon_{\alpha\beta} H_{13,\beta} + c_1^0 \epsilon_{\alpha\beta} H_{0\beta} - \epsilon_{\alpha\beta} \hat{H}_{1\beta}, \\ \dot{B}_{13} &= -\epsilon_{\alpha\beta} E_{1\beta,\alpha}, \quad \dot{B}_{1\alpha} = -\epsilon_{\alpha\beta} E_{13,\beta} - c_1^0 \epsilon_{\alpha\beta} E_{0\beta} + \epsilon_{\alpha\beta} \hat{E}_{1\beta}, \\ \bar{D}_{1\alpha,\alpha} &= c_1^0 \bar{D}_{03} - \hat{D}_{13}, \quad B_{1\alpha,\alpha} = c_1^0 B_{03} - \hat{B}_{13}, \end{aligned}\right\} \tag{10.5}$$

where $\epsilon_{\alpha\beta}$ is the alternating tensor with values 0, ± 1, and $\hat{D}_{03}$, $\hat{D}_{13}$, $\hat{B}_{03}$, $\hat{B}_{13}$, $\hat{E}_{0\alpha}$, $\hat{E}_{1\alpha}$, $\hat{H}_{0\alpha}$, $\hat{H}_{1\alpha}$, are given by (6.12) and (6.19).

The constitutive equations that are used with equations (10.2)–(10.5) are obtained from (7.4), (6.25)–(6.27) and (7.6), (7.7). These are

$$\left.\begin{aligned} N_{\alpha\beta} &= A_{\alpha\beta\lambda\mu} e_{\lambda\mu} + A_{\alpha\beta\lambda} \gamma_\lambda - P_{\alpha\beta}\theta - C^0_{\alpha\beta i} E_{0i} - F^0_{\alpha\beta i} H_{0i}, \\ k_\alpha &= A_{\lambda\mu\alpha} e_{\lambda\mu} + \bar{A}_{\alpha\beta}\gamma_\beta - P_\alpha \theta - C^0_{\alpha i} E_{0i} - F^0_{\alpha i} H_{0i}, \\ M_{\alpha\beta} &= B_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} - Q_{\alpha\beta}\phi - \bar{C}^1_{\alpha\beta i} E_{1i} - \bar{F}_{\alpha\beta i} H_{1i}, \\ \rho\eta &= P_{\alpha\beta} e_{\alpha\beta} + P_\alpha \gamma_\alpha + P\theta - R^0_i E_{0i} - S^0_i H_{0i}, \\ \rho\eta_1 &= Q_{\alpha\beta}\kappa_{\alpha\beta} + Q\phi - \bar{R}^1_i E_{1i} - \bar{S}^1_i H_{1i}, \\ \bar{D}_{0i} &= C^0_{\alpha\beta i} e_{\alpha\beta} + C^0_{\alpha i}\gamma_\alpha - R^0_i \theta + L^{00}_{ij} E_{0j} + M^{00}_{ij} H_{0j}, \\ B_{0i} &= F^0_{\alpha\beta i} e_{\alpha\beta} + F^0_{\alpha i}\gamma_\alpha - S^0_i\theta + M^{00}_{ji} E_{0j} + N^{00}_{ij} H_{0j}, \\ \bar{D}_{1i} &= \bar{C}^1_{\alpha\beta i}\kappa_{\alpha\beta} - \bar{R}^1_i \phi + L^{11}_{ij} E_{1j} + M^{11}_{ij} H_{1j}, \\ B_{1i} &= \bar{F}^1_{\alpha\beta i}\kappa_{\alpha\beta} - \bar{S}^1_i\phi + M^{11}_{ji} E_{1j} + N^{11}_{ji} H_{1j}, \\ e_{\alpha\beta} &= \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}), \quad \gamma_\alpha = \delta_\alpha + u_{3,\alpha}, \quad \kappa_{\alpha\beta} = \delta_{\alpha,\beta}. \end{aligned}\right\} \tag{10.6}$$

Discussion of the propagation of plane waves is now a straightforward but algebraically lengthy procedure. It is also necessary to consider electromagnetic wave propagation in the free space surrounding the plate and to use appropriate conditions for the electromagnetic vectors at the surfaces $z = \pm\frac{1}{2}h$. These conditions would involve continuity of the components E_α, H_α, $\bar{D}_3$, B_3 of the three-dimensional vectors at $z = \pm\frac{1}{2}h$. We do not embark on a general discussion here but note a slightly simpler situation in which the moduli for the plate are much greater than the moduli for free space so that we may adopt the approximate surface conditions

$$\bar{D}_3 = 0, \quad B_3 = 0 \quad (z = \pm\tfrac{1}{2}h), \tag{10.7}$$

and hence

$$\hat{D}_{03} = 0, \quad \hat{D}_{13} = 0, \quad \hat{B}_{03} = 0, \quad \hat{B}_{13} = 0. \tag{10.8}$$

In view of the representation under case (*a*) in (6.19) and the use only of the vectors $\bar{\boldsymbol{D}}_0$, $\bar{\boldsymbol{D}}_1$, $\boldsymbol{B}_0$, $\boldsymbol{B}_1$ for the plate, from (10.7) we have

$$\bar{D}_{03} = 0, \quad \bar{D}_{13} = 0, \quad B_{03} = 0, \quad B_{13} = 0. \tag{10.9}$$

Also, it follows from (10.7) that the surface values of E_α, H_α satisfy the equations

$$\epsilon_{\alpha\beta} E_{\alpha,\beta} = 0, \quad \epsilon_{\alpha\beta} H_{\alpha,\beta} = 0 \quad (z = \pm\tfrac{1}{2}h) \tag{10.10}$$

so that

$$\epsilon_{\alpha\beta}\hat{E}_{0\alpha,\beta} = 0, \quad \epsilon_{\alpha\beta}\hat{E}_{1\alpha,\beta} = 0, \quad \epsilon_{\alpha\beta}\hat{H}_{0\alpha,\beta} = 0, \quad \epsilon_{\alpha\beta}\hat{H}_{1\alpha,\beta} = 0. \tag{10.11}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Further, from $(10.4)_{1,3}$, $(10.5)_{1,3}$ and (10.9), it is seen that

$$E_{0\alpha} = -\chi_{0,\alpha}, \quad E_{1\alpha} = -\chi_{1,\alpha}, \quad H_{0\alpha} = -\psi_{0,\alpha}, \quad H_{1\alpha} = -\psi_{1,\alpha}, \tag{10.12}$$

where χ_0, χ_1, ψ_0, ψ_1 are scalar functions of x_α, t. The system of equations (10.4) and (10.5) may now be replaced by equation (10.9), (10.11), (10.12), together with

$$\bar{D}_{0\alpha,\alpha} = 0, \quad B_{0\alpha,\alpha} = 0, \quad \bar{D}_{1\alpha,\alpha} = 0, \quad \bar{B}_{1\alpha,\alpha} = 0, \tag{10.13}$$

and

$$\left.\begin{aligned} \dot{\bar{D}}_{01} &= H_{03,2} - \hat{H}_{02}, \quad \dot{B}_{01} = -E_{03,2} + \hat{E}_{02}, \\ \dot{\bar{D}}_{11} &= H_{13,2} + c_1^0 H_{02} - \hat{H}_{12}, \quad \dot{B}_{11} = -E_{13,2} - c_1^0 E_{02} + \hat{E}_{12}. \end{aligned}\right\} \tag{10.14}$$

The basic differential equations for the plate now separate into two groups: The first group consists of fifteen equations, (10.2), (10.3), (10.6), (10.9), (10.12) and (10.13), for the fifteen variables u_α, u_3, δ_β, θ, ϕ, χ_0, χ_1, ψ_0, ψ_1, E_{03}, E_{13}, H_{03}, H_{13}, while the second group consists of eight equations, (10.11) and (10.14), for the remaining variables $\hat{E}_{0\alpha}$, $\hat{H}_{0\alpha}$, $\hat{E}_{1\alpha}$, $\hat{H}_{1\alpha}$. We leave aside detailed consideration of wave propagation and other special problems.

11. Piezoelectric crystal plates

Isothermal vibrations of piezoelectric crystal plates due to given applied surface potentials may be studied as a special case of the theory of the previous sections. Here we limit our attention to the partially restricted theory of § 7 in which thermal variables are omitted, and the only relevant mechanical equations of motion are given by (7.2). In piezoelectric theory, the magnetic fields H_{Ni} are absent from all constitutive equations so that $B_{Mi} = 0$. We limit the remaining electromagnetic variables to E_{0i}, E_{1i}, $\bar{D}_{0i}$, $\bar{D}_{1i}$, E_{23}, $\bar{D}_{23}$ corresponding to the representation under case (a). Constitutive relations under case (a) are then given by $(7.4)_{1,2,3,6}$, in which the coefficients are given by (7.6), (7.7), (6.25), (6.26) and (6.27). In these equations, the terms involving the temperatures θ, ϕ, the magnetic terms H_{Mi} and the electric terms E_{21}, E_{22}, E_{Mi} ($M = 3, 4, \ldots$), as well as their corresponding coefficients, are omitted. Thus,

$$\left.\begin{aligned} N_{\alpha\beta} &= A_{\alpha\beta\lambda\mu} e_{\lambda\mu} + A_{\alpha\beta\lambda}\gamma_\lambda - C^0_{\alpha\beta i} E_{Mi}, \quad k_\alpha = A_{\lambda\mu\alpha} e_{\lambda\mu} + \bar{A}_{\alpha\beta}\gamma_\beta - C^0_{\alpha i} E_{Mi}, \\ M_{\alpha\beta} &= B_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} - \bar{C}^1_{\alpha\beta i} E_{Mi}, \quad \bar{D}_{0i} = C^0_{\alpha\beta i} e_{\alpha\beta} + C^0_{\alpha i}\gamma_\alpha + L^{00}_{ij} E_{0j}, \\ \bar{D}_{1i} &= \bar{C}^1_{\alpha\beta i}\kappa_{\alpha\beta} + L^{11}_{ij} E_{1j}, \quad \bar{D}_{23} = L^{22}_{33} E_{23}. \end{aligned}\right\} \tag{11.1}$$

By (6.11), (6.12) and (6.19), the appropriate electromagnetic field equations in this case are:

$$\left.\begin{aligned} \bar{D}_{0\alpha,\alpha} &= -h^{-\frac{1}{2}}[\bar{D}_3]^{\frac{1}{2}h}_{-\frac{1}{2}h}, \\ \bar{D}_{1\alpha,\alpha} &= c_1^0\{\bar{D}_{03} - h^{-\frac{1}{2}}[z\bar{D}_3]^{\frac{1}{2}h}_{-\frac{1}{2}h}\}, \end{aligned}\right\} \tag{11.2}$$

$$\left.\begin{aligned} \epsilon_{\alpha\beta} E_{0\beta,\alpha} &= 0, \quad \epsilon_{\alpha\beta} E_{1\beta,\alpha} = 0, \\ E_{03,\alpha} &= h^{-\frac{1}{2}}[E_\alpha]^{\frac{1}{2}h}_{-\frac{1}{2}h}, \\ E_{13,\alpha} &= c_1^0\{h^{-\frac{1}{2}}[zE_\alpha]^{\frac{1}{2}h}_{-\frac{1}{2}h} - E_{0\alpha}\}, \\ E_{23,\alpha} &= 5^{\frac{1}{2}}\{h^{-\frac{1}{2}}[E_\alpha]^{\frac{1}{2}h}_{-\frac{1}{2}h} - c_1^0 E_{1\alpha}\}, \end{aligned}\right\} \tag{11.3}$$

where $c_1^0 = 2.3^{\frac{1}{2}}/h$.

From $(11.3)_{1,2}$ it follows that

$$E_{0\alpha} = -\chi_{0,\alpha}, \quad E_{1\alpha} = -\chi_{1,\alpha}, \tag{11.4}$$

where χ_0, χ_1 are two-dimensional potential functions, and from $(11.3)_{3,4,5}$ we have

$$\left.\begin{aligned} [E_\alpha]_{-\frac{1}{2}h}^{\frac{1}{2}h} = -\Phi_{0,\alpha}, \quad [zE_\alpha]_{-\frac{1}{2}h}^{\frac{1}{2}h} = -\Phi_{1,\alpha}, \\ E_{03} = -h^{-\frac{1}{2}}\Phi_0, \quad E_{13} = c_1^0(\chi_0 - h^{-\frac{1}{2}}\Phi_1), \\ E_{23} = 5^{\frac{1}{2}}(c_1^0\chi_1 - h^{-\frac{1}{2}}\Phi_0), \end{aligned}\right\} \tag{11.5}$$

where Φ_0, Φ_1 are related to the applied voltages over the major surfaces $z = \pm\frac{1}{2}h$ of the plate, which are coated with electrodes. Within the framework of the present theory we may use the representation

$$\bar{D}_3 = h^{-\frac{1}{2}}\{\bar{D}_{03} + c_1^0 z\bar{D}_{13} + \tfrac{1}{2}5^{\frac{1}{2}}(12z^2/h^2 - 1)\,\bar{D}_{23})\} \tag{11.6}$$

so that in (11.2) we use

$$[\bar{D}_3]_{-\frac{1}{2}h}^{\frac{1}{2}h} = h^{\frac{1}{2}}c_1^0\,\bar{D}_{13}, \quad [z\bar{D}_3]_{-\frac{1}{2}h}^{\frac{1}{2}h} = h^{\frac{1}{2}}(\bar{D}_{03} + 5^{\frac{1}{2}}\bar{D}_{23}). \tag{11.7}$$

Also, at the surface $z = -\frac{1}{2}h$,

$$\bar{D}_{3|-\frac{1}{2}h} = h^{-\frac{1}{2}}(\bar{D}_{03} - \tfrac{1}{2}hc_1^0\bar{D}_{13} + 5^{\frac{1}{2}}\bar{D}_{23}). \tag{11.8}$$

The equations obtained here for piezoelectric plates are somewhat similar to those used by Tiersten & Mindlin (1962) and Tiersten (1969), although the basis of the present theory is quite different. These authors derived their equations from the three-dimensional equations of linear piezoelasticity with the help of expansion methods due to Cauchy and Poisson and the variational method of Kirchhoff, together with the introduction of correction factors involving the thickness-shear strains. We refer readers to these authors for a variety of applications to particular problems.

12. Alternative representation for plate theory

For some types of plate problems, depending on the nature of the surface conditions on the faces $z = \pm\frac{1}{2}h$, it is more appropriate to regard the response characterization of the medium in the context of the symmetry restrictions discussed under case (*b*) in (6.20). Again, we use the partially restricted theory of § 7 in the absence of body forces and also omit temperature effects. Assuming that the major surfaces of the plate are free from applied stresses, from (7.2) we have equations of motion

$$\rho\ddot{u}_i = N_{i\alpha,\alpha}, \quad \rho y^{11}\ddot{\delta}_\alpha = -k_\alpha + M_{\alpha\beta,\beta}, \quad k_\alpha = N_{3\alpha}, \quad \rho = \rho^* h, \quad y^{11} = \tfrac{1}{12}h^2. \tag{12.1}$$

For electromagnetic fields we consider only

$$E_{03}, E_{1i}, H_{0\alpha}, H_{1i}, B_{0\alpha}, B_{1i}, \bar{D}_{03}, \bar{D}_{1i}, \tag{12.2}$$

which, from (6.11) and (6.12), satisfy, in case (*b*), electromagnetic equations

$$\left.\begin{aligned} B_{0\alpha,\alpha} = -\hat{B}_{03}, \quad B_{1\alpha,\alpha} = -(\pi/h)\,B_{13} - \hat{B}_{13}, \quad \bar{D}_{1\alpha,\alpha} = (\pi/h)\,\bar{D}_{13}, \\ \dot{B}_{0\alpha} = -\epsilon_{\alpha\beta}E_{03,\beta} + \epsilon_{\alpha\beta}\hat{E}_{0\beta}, \quad \dot{B}_{13} = -\epsilon_{\alpha\beta}E_{1\beta,\alpha}, \\ \dot{B}_{1\alpha} = -\epsilon_{\alpha\beta}E_{13,\beta} + (\pi/h)\,\epsilon_{\alpha\beta}E_{1\beta} + \epsilon_{\alpha\beta}\hat{E}_{1\beta}, \\ \dot{\bar{D}}_{03} = \epsilon_{\alpha\beta}H_{0\beta,\alpha}, \quad \dot{\bar{D}}_{13} = \epsilon_{\alpha\beta}H_{1\beta,\alpha}, \\ \dot{\bar{D}}_{1\alpha} = \epsilon_{\alpha\beta}H_{13,\beta} + (\pi/h)\,\epsilon_{\alpha\beta}H_{1\beta}, \end{aligned}\right\} \tag{12.3}$$

where
$$\left.\begin{aligned} \hat{B}_{03} &= h^{-\frac{1}{2}}[B_3]^{\frac{1}{2}h}_{-\frac{1}{2}h}, \quad \hat{B}_{13} = -(2/h)^{\frac{1}{2}}[B_3 \sin \pi z/h]^{\frac{1}{2}h}_{-\frac{1}{2}h}, \\ \hat{E}_{0\beta} &= h^{-\frac{1}{2}}[E_\beta]^{\frac{1}{2}h}_{-\frac{1}{2}h}, \quad \hat{E}_{1\beta} = -(2/h)^{\frac{1}{2}}[E_\beta \sin \pi z/h]^{\frac{1}{2}h}_{-\frac{1}{2}h}. \end{aligned}\right\} \tag{12.4}$$

In the light of the values in (6.29) for case (b), (7.6) and (7.7), from (7.4) the appropriate constitutive equations are

$$\left.\begin{aligned} N_{\alpha\beta} &= A_{\alpha\beta\lambda\mu} e_{\lambda\mu} + A_{\alpha\beta\lambda} \gamma_\lambda - C^0_{\alpha\beta 3} E_{03} - C^1_{\beta\lambda} E_{1\lambda} - F^0_{\alpha\beta\lambda} H_{0\lambda} - F^1_{\alpha\beta 3} H_{13}, \\ k_\alpha &= A_{\lambda\mu\alpha} e_{\lambda\mu} + \bar{A}_{\alpha\beta} \gamma_\beta - C^0_{\alpha 3} E_{03} - C^1_{\alpha\lambda} E_{1\lambda} - F^0_{\alpha\lambda} H_{0\lambda} - F^1_{\alpha 3} H_{13}, \\ M_{\alpha\beta} &= B_{\alpha\beta\lambda\mu} \kappa_{\lambda\mu} - \bar{C}^1_{\alpha\beta 3} E_{13} - \bar{F}^1_{\alpha\beta\lambda} H_{1\lambda}, \\ \bar{D}_{03} &= C^0_{\alpha\beta 3} e_{\alpha\beta} + C^0_{\alpha 3} \gamma_\alpha + L^{01}_{3\alpha} E_{1\alpha} + L^{00}_{33} E_{03} + M^{00}_{3\alpha} H_{0\alpha} + M^{01}_{33} H_{13}, \\ \bar{D}_{1\alpha} &= C^1_{\lambda\mu\alpha} e_{\lambda\mu} + C^1_{\lambda\alpha} \gamma_\lambda + L^{11}_{\alpha\beta} E_{1\beta} + L^{10}_{\alpha 3} E_{03} + M^{10}_{\alpha\beta} H_{0\beta} + M^{11}_{\alpha 3} H_{13}, \\ \bar{D}_{13} &= \bar{C}^1_{\alpha\beta 3} \kappa_{\alpha\beta} + L^{11}_{33} E_{13} + M^{11}_{3\beta} H_{1\beta}, \\ B_{0\alpha} &= F^0_{\lambda\mu\alpha} e_{\lambda\mu} + F^0_{\lambda\alpha} \gamma_\lambda + M^{00}_{3\alpha} E_{03} + M^{10}_{\beta\alpha} E_{1\beta} + N^{00}_{\alpha\beta} H_{0\beta} + N^{01}_{\alpha 3} H_{13}, \\ B_{1\alpha} &= \bar{F}^1_{\lambda\mu\alpha} \kappa_{\lambda\mu} + F^1_{\lambda\alpha} \gamma_\lambda + M^{11}_{3\alpha} E_{13} + N^{11}_{\alpha\beta} H_{1\beta}, \\ B_{13} &= F^1_{\alpha\beta 3} e_{\alpha\beta} + F^1_{\alpha 3} \gamma_\alpha + M^{01}_{33} E_{03} + M^{11}_{\beta 3} E_{1\beta} + N^{10}_{3\beta} H_{0\beta} + N^{11}_{33} H_{13}. \end{aligned}\right\} \tag{12.5}$$

We consider one application in which the plate acts as an elastic wave guide, with the surfaces $z = \pm\frac{1}{2}h$ of the plate regarded as perfect conductors. For such a plate

$$B_3 = 0, \quad E_\beta = 0 \quad (z = \pm\tfrac{1}{2}h) \tag{12.6}$$

and the surface values of D_3, H_α are not known. The electromagnetic equations (12.3) are particularly suitable for this problem since in these equations surface values of D_3, H_α do not occur explicitly and

$$\hat{B}_{03} = 0, \quad \hat{B}_{13} = 0, \quad \hat{E}_{0\alpha} = 0, \quad \hat{E}_{1\alpha} = 0. \tag{12.7}$$

It is now straightforward to discuss wave propagation in such a plate with use of equations (12.1), (12.3), (12.5) and (12.7). Here we restrict attention to the special case of propagation in a large plate, which can be regarded as rigid. The relevant equations are then (12.3), (12.7) and (12.5) in which $e_{\alpha\beta} = 0$, $\gamma_\alpha = 0$, $\kappa_{\alpha\beta} = 0$. After removing an exponential factor, i.e. $\exp\{\mathrm{i}(m_\alpha x_\alpha + \omega t)\}$, we obtain

$$\left.\begin{aligned} (\omega M^{00}_{3\alpha} + \epsilon_{\alpha\beta} m_\beta) E_{03} + \omega M^{10}_{\beta\alpha} E_{1\beta} + \omega N^{00}_{\alpha\beta} H_{0\beta} + \omega N^{01}_{\alpha 3} H_{13} &= 0, \\ (\mathrm{i}\pi/h)\, \epsilon_{\alpha\beta} E_{1\beta} + (\omega M^{11}_{3\alpha} + \epsilon_{\alpha\beta} m_\beta) E_{13} + \omega N^{11}_{\alpha\beta} H_{1\beta} &= 0, \\ \omega M^{01}_{33} E_{03} + (\omega M^{11}_{\beta 3} + \epsilon_{\alpha\beta} m_\alpha) E_{1\beta} + \omega N^{10}_{3\beta} H_{0\beta} + \omega N^{11}_{33} H_{13} &= 0, \\ \omega L^{00}_{33} E_{03} + \omega L^{01}_{3\alpha} E_{1\alpha} + (\omega M^{00}_{3\alpha} + \epsilon_{\alpha\beta} m_\beta) H_{0\alpha} + \omega M^{01}_{33} H_{13} &= 0, \\ \omega L^{10}_{\alpha 3} E_{03} + \omega L^{11}_{\alpha\beta} E_{1\beta} + \omega M^{10}_{\alpha\beta} H_{0\beta} + (\mathrm{i}\pi/h)\, \epsilon_{\alpha\beta} H_{1\beta} + (\omega M^{11}_{\alpha 3} - \epsilon_{\alpha\beta} m_\beta) H_{13} &= 0, \\ \omega L^{11}_{33} E_{13} + (\omega M^{11}_{3\beta} - \epsilon_{\alpha\beta} m_\alpha) H_{1\beta} &= 0. \end{aligned}\right\} \tag{12.8}$$

The condition for a non-zero solution leads, in a usual way, to a 9×9 determinantal equation for ω, which we do not discuss in detail here. We note that when the plate is isotropic, with a centre of symmetry, the determinantal equation yields the values

$$\mu\epsilon\omega^2 = m_\alpha m_\alpha, \quad \mu\epsilon\omega^2 = m_\alpha m_\alpha + \pi^2/h^2, \tag{12.9}$$

where ϵ, μ are the isotropic coefficients for the electric and magnetic displacement vectors. The values (12.9) are the lowest exact frequencies that would be obtained by an exact three-dimensional solution of the wave guide problem for an isotropic plate.

The theory of the present paper according to the representation under case (*b*) is also suitable for problems studied by using a piezoelectric approximation. In this case the following constitutive coefficients are zero:

$$\left.\begin{aligned} &F^0_{\alpha\beta\lambda} = 0, \quad F^1_{\alpha\beta 3} = 0, \quad F^0_{\alpha\lambda} = 0, \; F^1_{\alpha 3} = 0, \quad \bar{F}^1_{\alpha\beta\lambda} = 0,\\ &M^{00}_{3\alpha} = 0, \quad M^{01}_{33} = 0, \quad M^{10}_{\alpha\beta} = 0, \quad M^{11}_{\alpha 3} = 0, \quad M^{11}_{3\beta} = 0,\\ &N^{00}_{\alpha\beta} = 0, \quad N^{01}_{\alpha 3} = 0, \quad N^{11}_{\alpha\beta} = 0, \quad N^{10}_{3\beta} = 0, \quad N^{11}_{33} = 0, \end{aligned}\right\} \tag{12.10}$$

so that

$$B_{0\alpha} = 0, \quad B_{1\alpha} = 0, \quad B_{13} = 0. \tag{12.11}$$

The appropriate electromagnetic equations (12.3) with $\hat{B}_{03} = 0$, $\hat{B}_{13} = 0$, reduce to

$$E_{03,\beta} - \hat{E}_{0\beta} = 0, \quad \epsilon_{\alpha\beta} E_{1\beta,\alpha} = 0, \quad \epsilon_{\alpha\beta} E_{13,\beta} = (\pi/h)\,\epsilon_{\alpha\beta} E_{1\beta} + \epsilon_{\alpha\beta}\hat{E}_{1\beta}, \tag{12.12}$$

$$\bar{D}_{1\alpha,\alpha} = (\pi/h)\,\bar{D}_{13}. \tag{12.13}$$

Setting

$$\hat{E}_{0\alpha} = -\Phi_{0,\alpha}, \quad \hat{E}_{1\alpha} = -\Phi_{1,\alpha}, \tag{12.14}$$

where Φ_0, Φ_1 are related to the applied voltages over the major surfaces $z = \pm\frac{1}{2}h$ of the plate (distinct from the corresponding quantities in (11.5)), it follows from (12.12) that

$$E_{03} = -\Phi_0, \quad E_{1\alpha} = -\chi_{,\alpha}, \quad E_{13} = -(\pi/h)\,\chi - \Phi_1, \tag{12.15}$$

where χ is a two-dimensional potential. Given the values of Φ_0, Φ_1, we have field equations (12.1) and (12.13) for the kinematic variables and χ. In such problems we usually need the value of the normal component $\bar{D}_3$ of the electric displacement vector at the surfaces of the plate. This can be found from the representation

$$\bar{D}_3 = h^{-\frac{1}{2}}\bar{D}_{03} - (2/h)^{\frac{1}{2}}\,\bar{D}_{13}\sin(\pi z/h). \tag{12.16}$$

More generality, which is needed for some types of piezoelectric plate problems, can be achieved by retaining more vectors representing electrical effects and by using a Cosserat surface $\mathscr{C}_P$ for $P > 1$. If we also interpret the mechanical part of the theory with the help of trigonometric instead of polynomial representations, then we obtain equations similar to those effected by a different procedure in Bugdayci & Bogy (1981), where some specific applications are also discussed.

13. Circular cylindrical membrane

The effect of electromagnetic fields on the surface properties of either a part of a non-conducting circular cylindrical shell or of a closed circular cylindrical shell can be studied as a special case of the general membrane theory of § 9. We consider a complete circular cylindrical shell of radius a under isothermal conditions that is free from body forces and has no applied stresses on its major surfaces. We restrict discussion to linear piezoelectric membranes and to electromagnetic variables corresponding to the representation under case (*b*), and consider only electromagnetic fields

$$E_{03}, \; E_{1i}, \; \bar{D}^3_0, \; \bar{D}^i_1. \tag{13.1}$$

From (9.9), the piezoelectric field equations are

$$E_{03,\beta} - \hat{E}_{0\beta} = 0, \quad \epsilon^{\alpha\beta} E_{1\beta,\alpha} = 0, \quad E_{13,\beta} = (\pi/h)\,E_{1\beta} + \hat{E}_{1\beta}, \quad \bar{D}^{\alpha}_{1|\alpha} = (\pi/h)\,\bar{D}^3_1. \tag{13.2}$$

With

$$\hat{E}_{0\alpha} = -\Phi_{0,\alpha}, \quad \hat{E}_{1\alpha} = -\Phi_{1,\alpha}, \tag{13.3}$$

where Φ_0, Φ_1 are related to the applied voltages over the major surfaces $z = \pm\frac{1}{2}h$ of the membrane, it follows that

$$E_{03} = -\Phi_0, \quad E_{1\alpha} = -\chi_{,\alpha}, \quad E_{13} = -(\pi/h)\,\chi - \Phi_1, \tag{13.4}$$

where χ is a surface potential.

The surface of the shell is given by the position vector

$$\boldsymbol{r} = \boldsymbol{e}_1 a\cos\theta + \boldsymbol{e}_2 a\sin\theta + \boldsymbol{e}_3 z, \tag{13.5}$$

where $\boldsymbol{e}_i$ are constant orthonormal vectors and $0 \leqslant \theta \leqslant 2\pi$. Then, selecting $\theta^1 = a\theta$, $\theta^2 = z$, we have

$$\left.\begin{aligned} A_{11} = A^{11} = A_{22} = A^{22} = 1, \quad A_{12} = A^{12} = 0,\\ \bar{B}_{11} = -a^{-1}, \quad \bar{B}_{12} = \bar{B}_{22} = 0, \end{aligned}\right\} \tag{13.6}$$

where $\bar{B}_{\alpha\beta}$ is the curvature tensor of the membrane surface. The equations of motion (9.7) reduce to

$$N^{\alpha\beta}{}_{,\beta} = \rho\ddot{u}^\alpha, \quad -N^{11}/a = \rho\ddot{u}^3, \tag{13.7}$$

where covariant differentiation now becomes partial differentiation with respect to θ^α. Also, from (9.6)

$$e_{11} = u_{1,1} + u_3/a, \quad e_{22} = u_{2,2}, \quad e_{12} = \tfrac{1}{2}(u_{1,2} + u_{2,1}). \tag{13.8}$$

In view of (13.6) we note that upper and lower indices denote the same values so that, for example,

$$u^1 = u_1, \quad u^2 = u_2, \quad u^3 = u_3. \tag{13.9}$$

The constitutive relations for $N^{\alpha\beta}$, $\bar{D}_0^3$, $\bar{D}_1^i$ are given by $(9.12)_{1,4}$ with the θ, H_{Mi} variables omitted. Values of the coefficients for case (b) are determined from the relevant expressions in (6.25), (6.29) and (9.15) if allowance is made for the appropriate tensor form for these coefficients. Thus,

$$\left.\begin{aligned} N^{\alpha\beta} &= A^{\alpha\beta\lambda\mu}e_{\lambda\mu} - C^{0\alpha\beta3}E_{03} - C^{1\alpha\beta\lambda}E_{1\lambda},\\ \bar{D}_0^3 &= C^{0\alpha\beta3}e_{\alpha\beta} + L^{0033}E_{03} + L^{013\lambda}E_{1\lambda},\\ \bar{D}_1^\alpha &= C^{1\lambda\mu\alpha}e_{\lambda\mu} + L^{10\alpha3}E_{03} + L^{11\alpha\beta}E_{1\beta},\\ \bar{D}_1^3 &= L^{1133}E_{13}, \end{aligned}\right\} \tag{13.10}$$

$$\left.\begin{aligned} A^{\alpha\beta\lambda\mu}A^*_{\lambda\mu\rho\nu} = \tfrac{1}{2}(\delta^\alpha_\rho\,\delta^\beta_\nu + \delta^\alpha_\nu\,\delta^\beta_\rho), \quad A^{\alpha\beta\lambda\mu}C^{*03}_{\lambda\mu} + C^{0\alpha\beta3} = 0, \quad A^{\alpha\beta\lambda\mu}C^{*1\rho}_{\lambda\mu} + C^{1\alpha\beta\rho} = 0,\\ L^{*0033} + L^{0033} = C^{*03}_{\alpha\beta}C^{0\alpha\beta3}, \quad L^{*10\lambda3} + L^{10\lambda3} = C^{*1\lambda}_{\alpha\beta}C^{0\alpha\beta3}, \quad L^{*11\lambda\mu} + L^{11\lambda\mu} = C^{*1\lambda}_{\alpha\beta}C^{1\alpha\beta\mu}, \end{aligned}\right\} \tag{13.11}$$

$$\left.\begin{aligned} A^*_{\alpha\beta\lambda\mu} = h^{-1}s_{\alpha\beta\lambda\mu}, \quad \rho = \rho^* h,\\ C^{*03}_{\alpha\beta} = h^{-\frac{1}{2}}k^{*3}_{\alpha\beta}, \quad C^{*1\rho}_{\lambda\mu} = (2^{\frac{3}{2}}/\pi h^{\frac{1}{2}})\,k^{*\rho}_{\lambda\mu},\\ L^{*0033} = f^{*33}, \quad L^{*013\lambda} = L^{*10\lambda3} = (2^{\frac{3}{2}}/\pi) f^{*\lambda3}, \quad L^{*11\lambda\mu} = f^{*\lambda\mu}. \end{aligned}\right\} \tag{13.12}$$

14. A rigid thin shell as a wave guide

The problem of a stationary rigid shell regarded as a wave guide, in which temperature effects are also disregarded, can be considered as a special case of the theory of §§ 2–4. Only electromagnetic variables will now appear in the theory and we consider only linear constitutive equations, interpreting the theory according to the representations (B 13) or case (b) in (6.20). Then the electromagnetic field equations (3.17)–(3.20) with $E_M = 0$, $\boldsymbol{J}_M = \boldsymbol{0}$ become

$$\left.\begin{aligned} A^{-\frac{1}{2}}(B^\alpha_M A^{\frac{1}{2}})_{,\alpha} = -(\pi M/h)\,B^3_M - [\chi_M(z)\,B^3 G^{\frac{1}{2}}/A^{\frac{1}{2}}]^{\frac{1}{2}h}_{-\frac{1}{2}h}, \quad A^{-\frac{1}{2}}(\bar{D}^\alpha_M A^{\frac{1}{2}})_{,\alpha} = (\pi M/h)\,\bar{D}^3_M,\\ \dot{B}^3_M = -\epsilon^{\alpha\beta}E_{M\beta,\alpha}, \quad \dot{B}^\alpha_M = -\epsilon^{\alpha\beta}\{E_{M3,\beta} - (\pi M/h)\,E_{M\beta} - [\chi_M(z)\,E_\beta]^{\frac{1}{2}h}_{-\frac{1}{2}h}\},\\ \dot{\bar{D}}^3_M = \epsilon^{\alpha\beta}H_{M\beta,\alpha}, \quad \dot{\bar{D}}^\alpha_M = \epsilon^{\alpha\beta}\{H_{M3,\beta} + (\pi M/h)\,H_{M\beta}\}, \end{aligned}\right\} \tag{14.1}$$

where $\epsilon^{\alpha\beta}$ is the surface alternating tensor. The linear constitutive equations are

$$\bar{D}^i_M = \sum_{N=0}^{L} (L^{MNij}E_{Nj} + M^{MNij}H_{Nj}), \quad B^i_M = \sum_{N=0}^{L} (M^{NMji}E_{Nj} + N^{MNij}H_{Nj}), \tag{14.2}$$

where the constitutive coefficients may be expressed in terms of the three-dimensional coefficients f^{rs}, g^{rs}, h^{rs} in (D 8) of Appendix D. Here we add the further restriction that the shell is thin compared with the minimum radius of curvature at any point of the material surface $\mathscr{S}$ identified with the middle surface of the rigid shell. With this assumption, the constitutive coefficients reduce to the following values, in which f^{rs}, g^{rs}, h^{rs} are evaluated on the middle surface of the shell:

$$\left.\begin{array}{l}
L^{M0\alpha\beta} = L^{0N\alpha\beta} = 0 \quad (M, N = 0, 1), \quad L^{MN\alpha\beta} = f^{\alpha\beta}\delta_{MN} \quad (M, N \geqslant 1), \\
L^{0N\alpha 3} = 0 \quad (N = 0, 1, \ldots), \quad L^{MM\alpha 3} = 0 \quad (M = 0, 1, \ldots), \\
L^{M0\alpha 3} = \dfrac{2^{\frac{1}{2}}}{\pi M}\{1-(-1)^M\}f^{\alpha 3} \quad (M \geqslant 1), \quad L^{MN\alpha 3} = \dfrac{2M}{\pi}\left\{\dfrac{1-(-1)^{M+N}}{M^2-N^2}\right\}f^{\alpha 3} \\
\hfill (M \neq N,\ M, N \geqslant 1), \\
L^{MN33} = f^{33}\delta_{MN} \quad (M, N = 0, 1, \ldots), \\
M^{0N\alpha\beta} = 0 \quad (N = 0, 1, \ldots), \quad M^{MM\alpha\beta} = 0 \quad (M = 0, 1, \ldots), \\
M^{M0\alpha\beta} = \dfrac{2^{\frac{1}{2}}}{\pi M}\{1-(-1)^M\}h^{\alpha\beta} \quad (M \geqslant 1), \quad M^{MN\alpha\beta} = \dfrac{2M}{\pi}\left\{\dfrac{1-(-1)^{M+N}}{M^2-N^2}\right\}h^{\alpha\beta} \\
\hfill (M \neq N,\ M, N \geqslant 1), \\
M^{0N\alpha 3} = M^{M0\alpha 3} = 0 \quad (M, N = 0, 1, \ldots), \\
M^{MN\alpha 3} = h^{\alpha 3}\delta_{MN} \quad (M, N \geqslant 1), \quad M^{MN3\alpha} = h^{3\alpha}\delta_{MN} \quad (M, N \geqslant 0), \\
M^{M033} = 0 \quad (M = 0, 1, \ldots), \quad M^{MM33} = 0 \quad (M = 0, 1, \ldots), \\
M^{0N33} = \dfrac{2^{\frac{1}{2}}}{\pi N}\{1-(-1)^N\}h^{33} \quad (N \geqslant 1), \quad M^{MN33} = \dfrac{2N}{\pi}\left\{\dfrac{1-(-1)^{M+N}}{N^2-M^2}\right\}h^{33} \\
\hfill (M \neq N,\ M, N \geqslant 1), \\
N^{MN\alpha\beta} = g^{\alpha\beta}\delta_{MN} \quad (M, N = 0, 1, \ldots), \\
N^{M0\alpha 3} = 0 \quad (M = 0, 1, \ldots), \quad N^{MM\alpha 3} = 0 \quad (M = 0, 1, \ldots), \\
N^{0N\alpha 3} = \dfrac{2^{\frac{1}{2}}}{\pi N}\{1-(-1)^N\}g^{\alpha 3} \quad (N \geqslant 1), \quad N^{MN\alpha 3} = \dfrac{2N}{\pi}\left\{\dfrac{1-(-1)^{M+N}}{N^2-M^2}\right\}g^{\alpha 3} \\
\hfill (M \neq N,\ M, N \geqslant 1), \\
N^{M033} = 0, \quad N^{0N33} = 0 \quad (M, N = 0, 1, \ldots), \quad N^{MN33} = g^{33}\delta_{MN} \quad (M, N \geqslant 1).
\end{array}\right\} \tag{14.3}$$

When the stationary rigid shell is regarded as a wave guide, the conditions on the major surfaces of the shell are

$$B^3 = 0, \quad E_\beta = 0 \quad (z = \pm\tfrac{1}{2}h) \tag{14.4}$$

so that the surface terms in (14.1) vanish. We consider, briefly, wave propagation in a long closed thin rigid circular cylindrical shell whose middle surface is specified by (13.5) with the surface metric tensor having the value in (13.6). The discussion is then limited to the electromagnetic components

$$E_{03}, E_{1i}, H_{0\alpha}, H_{1i}, \bar{D}^3_0, \bar{D}^i_1, B^\alpha_0, B^i_1 \tag{14.5}$$

and we assume that all quantities are proportional to $\exp[\mathrm{i}\{n\theta + mz + \omega t\}]$ where $n = 0, 1, 2, \ldots$. The frequency equation may then be found from (14.1), with $M = 0, 1$, (14.2) and (14.3) with

$M, N = 0, 0;\ 1, 0;\ 0, 1$. We merely note here that in the special case of an isotropic shell, the frequencies are given by

$$\mu\epsilon\omega^2 = m^2 + n^2/a^2, \quad \mu\epsilon\omega^2 = m^2 + n^2/a^2 + \pi^2/h^2, \tag{14.6}$$

where μ, ϵ are the isotropic coefficients.

Further values of the frequencies could be found by extending the number of electromagnetic variables used in (14.1)–(14.3) beyond those in (14.5).

The work of one of us (P. M. N.) was supported by the U.S. Office of Naval Research under contract N00014-75-C-0148, project NR 064–436 with the University of California, Berkeley (U.C.B.). During 1980, A.E.G. held a visiting appointment in U.C.B. and would like to acknowledge the support of a Leverhulme Fellowship for the period 1979–80.

Appendix A

This appendix contains a brief summary of the three-dimensional theory of electromagnetism of moving deformable media. In particular, all conservation laws and the associated local field equations are recorded in order to provide some background information for some aspects of the developments in §§ 2 and 3 of the paper.

Consider a body $\mathscr{B}$ consisting of particles X and let $\mathscr{B}_t$ be the configuration of $\mathscr{B}$ at time t bounded by a closed surface $\partial\mathscr{B}_t$. A motion of the body is defined by a sufficiently smooth vector function $\boldsymbol{\chi}$ which assigns to each X the place $\boldsymbol{r}^* = \boldsymbol{\chi}(X, t)$ in the configuration $\mathscr{B}_t$. Let $\mathscr{S}_t^*$ be a regular material surface in $\mathscr{B}_t$ with closed regular edges $\partial\mathscr{S}_t^*$ and let $\mathscr{P}_t^*$ be a regular material volume in $\mathscr{B}_t$ with closed regular boundary surface $\partial\mathscr{P}_t^*$ whose outward unit normal is $\boldsymbol{u}$. The electrodynamic and continuum balance laws are

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{S}_t^*} \boldsymbol{b}\cdot\mathrm{d}\boldsymbol{a} = -\int_{\partial\mathscr{S}_t^*} \boldsymbol{e}^*\cdot\mathrm{d}\boldsymbol{x}^*, \tag{A 1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\mathscr{S}_t^*} \bar{\boldsymbol{d}}\cdot\mathrm{d}\boldsymbol{a} = \int_{\partial\mathscr{S}_t^*} \boldsymbol{h}^*\cdot\mathrm{d}\boldsymbol{x}^* - \int_{\mathscr{S}_t^*} \boldsymbol{j}^*\cdot\mathrm{d}\boldsymbol{a}, \tag{A 2}$$

$$\int_{\partial\mathscr{P}_t^*} \boldsymbol{b}\cdot\mathrm{d}\boldsymbol{a} = 0, \quad \int_{\partial\mathscr{P}_t^*} \bar{\boldsymbol{d}}\cdot\mathrm{d}\boldsymbol{a} = \int_{\mathscr{P}_t^*} e\,\mathrm{d}v, \tag{A 3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_t^*} \rho^*\,\mathrm{d}v = 0, \tag{A 4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_t^*} \rho^*\boldsymbol{v}^*\,\mathrm{d}v = \int_{\mathscr{P}_t^*} \rho^*(\boldsymbol{f}^* + \boldsymbol{f}_\mathrm{e}^*)\,\mathrm{d}v + \int_{\partial\mathscr{P}_t^*} \boldsymbol{t}\,\mathrm{d}a, \tag{A 5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_t^*} \rho^*\boldsymbol{r}^*\times\boldsymbol{v}^*\,\mathrm{d}v = \int_{\mathscr{P}_t^*} \{\boldsymbol{r}^*\times(\boldsymbol{f}^* + \boldsymbol{f}_\mathrm{e}^*) + \boldsymbol{c}_\mathrm{e}^*\}\rho^*\,\mathrm{d}v + \int_{\partial\mathscr{P}_t^*} \boldsymbol{r}^*\times\boldsymbol{t}\,\mathrm{d}a, \tag{A 6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_t^*} \rho^*\eta^*\,\mathrm{d}v = \int_{\mathscr{P}_t^*} \rho^*(s^* + \xi^*)\,\mathrm{d}v - \int_{\partial\mathscr{P}_t^*} k^*\,\mathrm{d}a, \tag{A 7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_t^*} \rho^*(\epsilon^* + \tfrac{1}{2}\boldsymbol{v}^*\cdot\boldsymbol{v}^*)\,\mathrm{d}v = \int_{\mathscr{P}_t^*} \{r^* + (\boldsymbol{f}^* + \boldsymbol{f}_\mathrm{e}^*)\cdot\boldsymbol{v}^* + \tfrac{1}{2}\boldsymbol{c}_\mathrm{e}^*\cdot\mathrm{curl}^*\,\boldsymbol{v}^* + w_\mathrm{e}^*\}\rho^*\,\mathrm{d}v + \int_{\partial\mathscr{P}_t^*} (\boldsymbol{t}\cdot\boldsymbol{v}^* - h^*)\,\mathrm{d}a. \tag{A 8}$$

In equations (A 1)–(A 8) and in the configuration $\mathscr{B}_t$ at time t, ρ^* is density, $\boldsymbol{v}^* = \dot{\boldsymbol{\chi}}(X,t)$ is velocity, $\boldsymbol{f}^*$ is external body force density, $\boldsymbol{f}_e^*$ is body force density and $\boldsymbol{c}_e^*$ is body force couple density due to the electromagnetic field, $\boldsymbol{t}$ is surface traction across $\partial\mathscr{P}_t^*$, h^* is flux of heat and k^* is flux of entropy across $\partial\mathscr{P}_t^*$, ϵ^* is internal energy density, η^* is entropy density, r^* is external volume rate of supply of heat density, s^* is external volume rate of supply of entropy density, w_e^* is the volume rate of supply of electromagnetic energy density due to the electromagnetic fields, $\boldsymbol{e}$ is the electric field intensity, $\bar{\boldsymbol{d}}$ is the electric displacement vector, $\boldsymbol{h}$ is the magnetic field intensity (axial) vector, $\boldsymbol{b}$ is the magnetic induction (axial) vector, $\boldsymbol{j}$ is the current density, e is the free charge, and

$$\boldsymbol{e}^* = \boldsymbol{e}+\boldsymbol{v}^*\times\boldsymbol{b},\quad \boldsymbol{h}^* = \boldsymbol{h}-\boldsymbol{v}^*\times\bar{\boldsymbol{d}},\quad \boldsymbol{j}^* = \boldsymbol{j}-e\boldsymbol{v}^*. \tag{A 9}$$

Field equations that can be derived from (A 1)–(A 8) are

$$\left.\begin{aligned}\operatorname{curl}^*\boldsymbol{e}^* &= -(\dot{\boldsymbol{b}}+\boldsymbol{b}\operatorname{div}^*\boldsymbol{v}^*-\boldsymbol{L}^*\boldsymbol{b}),\quad \operatorname{div}^*\boldsymbol{b}=0,\\ \operatorname{curl}^*\boldsymbol{h}^* &= \boldsymbol{j}^*+\dot{\bar{\boldsymbol{d}}}+\bar{\boldsymbol{d}}\operatorname{div}^*\boldsymbol{v}^*-\boldsymbol{L}^*\bar{\boldsymbol{d}},\quad \operatorname{div}^*\bar{\boldsymbol{d}} = e,\end{aligned}\right\} \tag{A 10}$$

$$\left.\begin{aligned}&\dot{\rho}^*+\rho^*\operatorname{div}^*\boldsymbol{v}^*=0,\quad \rho^*\dot{\boldsymbol{v}}^*=\rho^*(\boldsymbol{f}^*+\boldsymbol{f}_e^*)+\operatorname{div}^*\boldsymbol{T},\quad \boldsymbol{t}=\boldsymbol{T}\boldsymbol{u},\\ &\rho^*\boldsymbol{\Gamma}_e^*+\boldsymbol{T}-\boldsymbol{T}^{\mathrm{T}}=\boldsymbol{0},\quad \rho^*\dot{\eta}^*=\rho^*(s^*+\xi^*)-\operatorname{div}^*\boldsymbol{p}^*,\quad k^*=\boldsymbol{p}^*\cdot\boldsymbol{u},\quad h^*=\boldsymbol{q}^*\cdot\boldsymbol{u},\\ &\boldsymbol{q}^*=\theta^*\boldsymbol{p}^*,\quad \rho^*r^*-\operatorname{div}^*\boldsymbol{q}^*-\rho^*\dot{\epsilon}^*+\rho^*w_e^*+\boldsymbol{T}\cdot\boldsymbol{L}^*+\tfrac{1}{2}\rho^*\boldsymbol{\Gamma}_e^*\cdot\boldsymbol{L}^*=0,\end{aligned}\right\} \tag{A 11}$$

where $\theta^* > 0$ is temperature and

$$\boldsymbol{L}^* = \partial\boldsymbol{v}^*/\partial\boldsymbol{r}^*,\quad \boldsymbol{\Gamma}_e^*\boldsymbol{z} = \boldsymbol{c}_e^*\times\boldsymbol{z} \tag{A 12}$$

for every vector $\boldsymbol{z}$. Also div*, curl* are the divergence and curl operators with respect to the position $\boldsymbol{x}^*$.

Following Hutter & van de Ven (1978), we also make use of equations (A 1)–(A 11) in a material description with respect to a reference configuration of the body $\mathscr{B}_R$ whose points are specified by the position vector $\boldsymbol{R}^*$ and whose motion is defined by $\boldsymbol{r}^* = \boldsymbol{\chi}^*(\boldsymbol{R}^*,t)$. Let $\mathscr{S}_R^*$ be a regular surface in $\mathscr{B}_R$ with closed edges $\partial\mathscr{S}_R^*$, and let $\mathscr{P}_R^*$ be a regular volume in $\mathscr{B}_R$ with a closed regular boundary surface $\partial\mathscr{P}_R^*$ whose outward unit normal is $\boldsymbol{u}_R$, corresponding, respectively, to $\mathscr{S}_t^*$, $\partial\mathscr{S}_t^*$, $\mathscr{P}_t^*$, $\partial\mathscr{P}_t^*$ in the configuration $\mathscr{B}_t$. Then, setting

$$\left.\begin{aligned}&\boldsymbol{F}^*=\partial\boldsymbol{\chi}^*/\partial\boldsymbol{X}^*,\quad \Gamma^*=\det\boldsymbol{F}^*>0,\quad \rho_R^*=\Gamma^*\rho^*,\\ &\boldsymbol{E}=\boldsymbol{F}^{*\mathrm{T}}\boldsymbol{e}^*,\quad \boldsymbol{H}=\boldsymbol{F}^{*\mathrm{T}}\boldsymbol{h}^*,\quad \bar{\boldsymbol{D}}=\Gamma^*\boldsymbol{F}^{*-1}\bar{\boldsymbol{d}},\quad \boldsymbol{B}=\Gamma^*\boldsymbol{F}^{*-1}\boldsymbol{b},\\ &\Gamma^*\boldsymbol{T}=\boldsymbol{T}_R\boldsymbol{F}^{*\mathrm{T}},\quad \boldsymbol{t}_R=\boldsymbol{T}_R\boldsymbol{u}_R,\quad E=\Gamma^*e,\quad \boldsymbol{J}=\Gamma^*\boldsymbol{F}^{*-1}\boldsymbol{j}^*,\\ &\Gamma^*\boldsymbol{p}^*=\boldsymbol{F}^*\boldsymbol{p}_R^*,\quad \Gamma^*\boldsymbol{q}^*=\boldsymbol{F}^*\boldsymbol{q}_R^*,\quad k_R^*=\boldsymbol{p}_R^*\cdot\boldsymbol{u}_R,\quad h_R^*=\boldsymbol{q}_R^*\cdot\boldsymbol{u}_R,\quad \boldsymbol{q}_R^*=\theta^*\boldsymbol{p}_R^*,\end{aligned}\right\} \tag{A 13}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{S}_R^*}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{A} = -\int_{\partial\mathscr{S}_R^*}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{X}^*, \tag{A 14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{S}_R^*}\bar{\boldsymbol{D}}\cdot\mathrm{d}\boldsymbol{A} = \int_{\partial\mathscr{S}_R^*}\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{X}^* - \int_{\mathscr{S}_R^*}\boldsymbol{J}\cdot\mathrm{d}\boldsymbol{A}, \tag{A 15}$$

$$\int_{\partial\mathscr{P}_R^*}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{A} = 0,\quad \int_{\partial\mathscr{P}_R^*}\bar{\boldsymbol{D}}\cdot\mathrm{d}\boldsymbol{A} = \int_{\mathscr{P}_R^*}E\,\mathrm{d}V, \tag{A 16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}_R^*}\rho_R^*\boldsymbol{v}^*\,\mathrm{d}V = \int_{\mathscr{P}_R^*}\rho_R^*(\boldsymbol{f}^*+\boldsymbol{f}_e^*)\,\mathrm{d}V + \int_{\partial\mathscr{P}_R^*}\boldsymbol{t}_R\,\mathrm{d}A, \tag{A 17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*_\mathrm{R}} \rho^*_\mathrm{R}\, \boldsymbol{r}^* \times \boldsymbol{v}^* \,\mathrm{d}V = \int_{\mathscr{P}^*_\mathrm{R}} \{\boldsymbol{r}^* \times (\boldsymbol{f}^* + \boldsymbol{f}^*_\mathrm{e}) + \boldsymbol{c}^*_\mathrm{e}\}\rho^*_\mathrm{R}\,\mathrm{d}V + \int_{\partial\mathscr{P}^*_\mathrm{R}} \boldsymbol{r}^* \times \boldsymbol{t}_\mathrm{R}\,\mathrm{d}A, \tag{A 18}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*_\mathrm{R}} \rho^*_\mathrm{R}\,\eta^*\,\mathrm{d}V = \int_{\mathscr{P}^*_\mathrm{R}} \rho^*_\mathrm{R}(s^* + \xi^*)\,\mathrm{d}V - \int_{\partial\mathscr{P}^*_\mathrm{R}} k^*_\mathrm{R}\,\mathrm{d}A, \tag{A 19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*_\mathrm{R}} \rho^*_\mathrm{R}(\epsilon^* + \tfrac{1}{2}\boldsymbol{v}^*\cdot\boldsymbol{v}^*)\,\mathrm{d}V = \int_{\mathscr{P}^*_\mathrm{R}} \{r^* + (\boldsymbol{f}^* + \boldsymbol{f}^*_\mathrm{e})\cdot\boldsymbol{v}^* + \tfrac{1}{2}\boldsymbol{c}^*_\mathrm{e}\cdot\mathrm{curl}^*\,\boldsymbol{v}^* + w^*_\mathrm{e}\}\rho^*_\mathrm{R}\,\mathrm{d}V + \int_{\partial\mathscr{P}^*_\mathrm{R}} (\boldsymbol{t}_\mathrm{R}\cdot\boldsymbol{v}^* - h^*_\mathrm{R})\,\mathrm{d}A. \tag{A 20}$$

Field equations that correspond to equations (A 14)–(A 20) are

$$\left.\begin{gathered}\mathrm{Curl}^*\,\boldsymbol{E} = -\dot{\boldsymbol{B}},\quad \mathrm{Curl}^*\,\boldsymbol{H} = \dot{\bar{\boldsymbol{D}}} + \boldsymbol{J},\quad \mathrm{Div}^*\,\boldsymbol{B} = 0,\quad \mathrm{Div}^*\,\bar{\boldsymbol{D}} = E,\\ \rho^*_\mathrm{R}\dot{\boldsymbol{v}}^* = \rho^*_\mathrm{R}(\boldsymbol{f}^* + \boldsymbol{f}^*_\mathrm{e}) + \mathrm{Div}^*\,\boldsymbol{T}_\mathrm{R},\quad \rho^*_\mathrm{R}\boldsymbol{\Gamma}^*_\mathrm{e} + \boldsymbol{T}_\mathrm{R}\boldsymbol{F}^{*\mathrm{T}} - \boldsymbol{F}^*\boldsymbol{T}^\mathrm{T}_\mathrm{R} = \boldsymbol{0},\\ \rho^*_\mathrm{R}\dot{\eta}^* = \rho^*_\mathrm{R}(s^* + \xi^*) - \mathrm{Div}^*\,\boldsymbol{p}^*_\mathrm{R},\\ \rho^*_\mathrm{R}r^* - \mathrm{Div}^*\,\boldsymbol{q}^*_\mathrm{R} - \rho^*_\mathrm{R}\dot{\epsilon}^* + \rho^*_\mathrm{R}w^*_\mathrm{e} + \boldsymbol{T}_\mathrm{R}\cdot\dot{\boldsymbol{F}}^* + \tfrac{1}{2}\rho^*_\mathrm{R}\boldsymbol{\Gamma}^*_\mathrm{e}\cdot\boldsymbol{W}^* = 0,\end{gathered}\right\} \tag{A 21}$$

where Div*, Curl* are divergence and curl operators with respect to $\boldsymbol{R}^*$ in the reference configuration.

In the development of shell theory from three-dimensional equations of classical continuum mechanics, it is convenient to introduce a system of curvilinear coordinates θ^i $(i = 1, 2, 3)$ in the reference configuration of the body and regard these as a convected system of coordinates throughout the motion. To provide some additional background information, consider a finite three-dimensional body embedded in a Euclidean 3-space and identify each material point (or particle) of the body by the convected coordinates θ^i. Further, let $\boldsymbol{r}^*$ be the position vector, from a fixed origin, of a typical particle in the present configuration of the body at time t; and, similarly, let $\boldsymbol{R}^*$ denote the position vector in a fixed reference configuration, which may be taken to be the initial configuration of the body. Then in the context of the three-dimensional theory, various kinematical results may be stated as†

$$\left.\begin{gathered}\boldsymbol{R}^* = \boldsymbol{R}^*(\theta^i),\quad \boldsymbol{r}^* = \boldsymbol{r}^*(\theta^i, t),\quad \boldsymbol{G}_i = \partial\boldsymbol{R}^*/\partial\theta^i,\quad \boldsymbol{g}_i = \partial\boldsymbol{r}^*/\partial\theta^i,\\ \boldsymbol{G}^i\cdot\boldsymbol{G}_j = \delta^i_j,\quad \boldsymbol{g}^i\cdot\boldsymbol{g}_j = \delta^i_j,\quad G_{ij} = \boldsymbol{G}_i\cdot\boldsymbol{G}_j,\quad G^{ij} = \boldsymbol{G}^i\cdot\boldsymbol{G}^j,\\ g_{ij} = \boldsymbol{g}_i\cdot\boldsymbol{g}_j,\quad g^{ij} = \boldsymbol{g}^i\cdot\boldsymbol{g}^j,\quad g^{\frac{1}{2}} = [\boldsymbol{g}_1\boldsymbol{g}_2\boldsymbol{g}_3],\quad G^{\frac{1}{2}} = [\boldsymbol{G}_1\boldsymbol{G}_2\boldsymbol{G}_3],\\ \boldsymbol{F}^* = \boldsymbol{g}_i\otimes\boldsymbol{G}^i,\quad \boldsymbol{g}_i = \boldsymbol{F}^*\boldsymbol{G}_i,\quad \boldsymbol{G}^i = \boldsymbol{F}^{*\mathrm{T}}\boldsymbol{g}^i,\\ \Gamma^* = \det\boldsymbol{F}^* = g^{\frac{1}{2}}/G^{\frac{1}{2}},\quad \boldsymbol{L}^* = \dot{\boldsymbol{g}}_i\otimes\boldsymbol{g}^i,\end{gathered}\right\} \tag{A 22}$$

where $\boldsymbol{g}_i$, $\boldsymbol{g}^i$ are covariant and contravariant base vectors, respectively, g_{ij} and g^{ij} are covariant and contravariant metric tensors, respectively, in the configuration at time t, and δ^i_j is the Kronecker delta. Corresponding quantities for the reference configuration are $\boldsymbol{G}_i$, $\boldsymbol{G}^i$, G_{ij}, G^{ij}. Also‡

$$\left.\begin{gathered}\boldsymbol{t}^i = \boldsymbol{T}\boldsymbol{g}^i,\quad \boldsymbol{t}^i_\mathrm{R} = \boldsymbol{T}_\mathrm{R}\boldsymbol{G}^i,\quad g^{\frac{1}{2}}\boldsymbol{t}^i = G^{\frac{1}{2}}\boldsymbol{t}^i_\mathrm{R},\quad \boldsymbol{T} = \boldsymbol{t}^i\otimes\boldsymbol{g}_i,\quad \boldsymbol{T}_\mathrm{R} = \boldsymbol{t}^i_\mathrm{R}\otimes\boldsymbol{G}_i,\\ g^{\frac{1}{2}}\,\mathrm{div}^*\,\boldsymbol{T} = (g^{\frac{1}{2}}\boldsymbol{t}^i)_{,i} = (G^{\frac{1}{2}}\boldsymbol{t}^i_\mathrm{R})_{,i} = G^{\frac{1}{2}}\,\mathrm{Div}^*\,\boldsymbol{T}_\mathrm{R},\\ g^{\frac{1}{2}}\boldsymbol{p}^*\cdot\boldsymbol{g}^i = G^{\frac{1}{2}}\boldsymbol{p}^*_\mathrm{R}\cdot\boldsymbol{G}^i,\quad g^{\frac{1}{2}}\,\mathrm{div}^*\,\boldsymbol{p}^* = (g^{\frac{1}{2}}\boldsymbol{p}^*\cdot\boldsymbol{g}^i)_{,i} = (G^{\frac{1}{2}}\boldsymbol{p}^*_\mathrm{R}\cdot\boldsymbol{G}^i)_{,i} = G^{\frac{1}{2}}\,\mathrm{Div}^*\,\boldsymbol{p}^*_\mathrm{R},\end{gathered}\right\} \tag{A 23}$$

† Some of the notation in (A 22) differs from the corresponding symbols used previously (Naghdi 1972; Green & Naghdi 1976, 1979).

‡ The notation $g^{\frac{1}{2}}\boldsymbol{t}^i$ in (A 23)$_3$ corresponds to $\boldsymbol{T}^i$ (or $\boldsymbol{T}_i$) used previously (for example, in Naghdi 1972; Green & Naghdi 1976).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

a comma denoting partial derivative with respect to θ^i. The electromagnetic vectors may be expressed in component forms by the relations

$$\left.\begin{aligned} &\boldsymbol{e}^* = e_i^* \boldsymbol{g}^i, \quad \boldsymbol{h}^* = h_i^* \boldsymbol{g}^i, \quad \boldsymbol{E} = E_i \boldsymbol{G}^i, \quad \boldsymbol{H} = H_i \boldsymbol{G}^i, \quad \bar{\boldsymbol{d}} = \bar{d}^i \boldsymbol{g}_i, \\ &\boldsymbol{b} = b^i \boldsymbol{g}_i, \quad \bar{\boldsymbol{D}} = \bar{D}^i \boldsymbol{G}_i, \quad \boldsymbol{B} = B^i \boldsymbol{G}_i, \quad \boldsymbol{j}^* = j^{*i} \boldsymbol{g}_i, \quad \boldsymbol{J} = J^i \boldsymbol{G}_i, \end{aligned}\right\} \tag{A 24}$$

so that, with the help of (A 13), we have

$$E_i = e_i^*, \quad H_i = h_i^*, \quad G^{\frac{1}{2}} \bar{D}^i = g^{\frac{1}{2}} \bar{d}^i, \quad G^{\frac{1}{2}} B^i = g^{\frac{1}{2}} b^i, \quad G^{\frac{1}{2}} J^i = g^{\frac{1}{2}} j^{*i}, \quad G^{\frac{1}{2}} E = g^{\frac{1}{2}} e. \tag{A 25}$$

To complete the three-dimensional theory we must specify values for $\boldsymbol{\Gamma}_{\mathrm{e}}^*$ (or $\boldsymbol{c}_{\mathrm{e}}^*$), $\boldsymbol{f}_{\mathrm{e}}^*$ and w_{e}^*. Here we record values that are a slight modification of those derived by Hutter and van der Ven (1978):

$$\left.\begin{aligned} \rho^* \boldsymbol{\Gamma}_{\mathrm{e}}^* &= \boldsymbol{T}_{\mathrm{e}} - \boldsymbol{T}_{\mathrm{e}}^{\mathrm{T}}, \quad \rho^* \boldsymbol{c}_{\mathrm{e}}^* = \boldsymbol{g}_i \times \boldsymbol{t}_{\mathrm{e}}^i, \quad \boldsymbol{t}_{\mathrm{e}}^i = \boldsymbol{T}_{\mathrm{e}} \boldsymbol{g}^i, \\ \boldsymbol{T}_{\mathrm{e}} &= \boldsymbol{e}^* \otimes \bar{\boldsymbol{d}} + \boldsymbol{h}^* \otimes \boldsymbol{b} - \tfrac{1}{2}(\epsilon_0 \boldsymbol{e}^* \cdot \boldsymbol{e}^* + \mu_0 \boldsymbol{h}^* \cdot \boldsymbol{h}^*) \boldsymbol{I}, \\ \rho^* \boldsymbol{f}_{\mathrm{e}}^* &= e \boldsymbol{e}^* + \boldsymbol{j}^* \times \boldsymbol{b} + (\bar{\boldsymbol{d}} - \epsilon_0 \boldsymbol{e}^*) \cdot \boldsymbol{\nabla} \boldsymbol{e}^* + (\boldsymbol{b} - \mu_0 \boldsymbol{h}^*) \cdot \boldsymbol{\nabla} \boldsymbol{h}^* + \dot{\overline{\bar{\boldsymbol{d}} \times \boldsymbol{b}}} \\ &\qquad + (\bar{\boldsymbol{d}} \times \boldsymbol{b}) \operatorname{div}^* \boldsymbol{v}^* + \boldsymbol{L}^{*\mathrm{T}} (\bar{\boldsymbol{d}} \times \boldsymbol{b}), \\ \rho^* w_{\mathrm{e}}^* + \tfrac{1}{2} \rho^* \boldsymbol{\Gamma}_{\mathrm{e}}^* \cdot \boldsymbol{L}^* &= \boldsymbol{T}_{\mathrm{e}} \cdot \boldsymbol{L}^* + \boldsymbol{e}^* \cdot \boldsymbol{j}^* + \boldsymbol{e}^* \cdot (\dot{\bar{\boldsymbol{d}}} + \bar{\boldsymbol{d}} \operatorname{div}^* \boldsymbol{v}^* - \boldsymbol{L}^* \bar{\boldsymbol{d}}) \\ &\qquad + \boldsymbol{h}^* \cdot (\dot{\boldsymbol{b}} + \boldsymbol{b} \operatorname{div}^* \boldsymbol{v}^* - \boldsymbol{L}^* \boldsymbol{b}), \\ \rho_{\mathrm{R}}^* w_{\mathrm{e}}^* + \tfrac{1}{2} \rho_{\mathrm{R}}^* \boldsymbol{\Gamma}_{\mathrm{e}}^* \cdot \boldsymbol{L}^* &= \boldsymbol{T}_{\mathrm{Re}} \cdot \dot{\boldsymbol{F}}^* + \boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \dot{\bar{\boldsymbol{D}}} + \boldsymbol{H} \cdot \dot{\boldsymbol{B}}, \quad \boldsymbol{\Gamma}^* \boldsymbol{T}_{\mathrm{e}} = \boldsymbol{T}_{\mathrm{Re}} \boldsymbol{F}^{*\mathrm{T}}, \end{aligned}\right\} \tag{A 26}$$

where ϵ_0, μ_0 are the electromagnetic coefficients for a vacuum.

Appendix B

The purpose of this Appendix is to provide some formulae that arise in the development of shell theory from the three-dimensional equations of classical continuum mechanics. Formulae of the type obtained here have been given previously in the context of thermomechanical theory (see Naghdi 1972; Green & Naghdi 1976, 1978). However, in this Appendix, we provide slightly more general formulae, which include results from electromagnetism. For our present purpose, it is convenient to adopt the notation $\theta^3 = z$ so that the convected coordinate system described in Appendix A (preceding equations (A 22)) can be designated as $\theta^i = (\theta^\alpha, z)$, $\alpha = 1, 2$. Recalling also from Appendix A the notation $\boldsymbol{r}^*$ and $\boldsymbol{R}^*$ for material points in the current and reference configurations, respectively, we suppose now that the position vectors $\boldsymbol{r}^*$ and $\boldsymbol{R}^*$, as well as the temperature field θ^*, may be specified as

$$\left.\begin{aligned} &\boldsymbol{R}^* = \boldsymbol{R} + \sum_{N=1}^{P} \lambda_N(z) \boldsymbol{D}_N, \quad \boldsymbol{R} = \boldsymbol{R}(\theta^\alpha), \quad \boldsymbol{D}_N = \boldsymbol{D}_N(\theta^\alpha), \quad \theta^3 = z, \\ &\boldsymbol{r}^* = \boldsymbol{r} + \sum_{N=1}^{P} \lambda_N(z) \boldsymbol{d}_N, \quad \theta^* = \theta + \sum_{M=1}^{K} \mu_M(z) \theta_M, \\ &\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha, t), \quad \boldsymbol{d}_N = \boldsymbol{d}_N(\theta^\alpha, t), \quad \theta = \theta(\theta^\alpha, t), \quad \theta_M = \theta_M(\theta^\alpha, t) \end{aligned}\right\} \tag{B 1}$$

in the region $z_1 \leqslant z \leqslant z_2$, where z_1, z_2 are constants and where Greek indices take the values 1, 2. The major surfaces of the shell correspond to $z = z_1$, $z = z_2$. Previously, $\lambda_N(z)$, $\mu_N(z)$ have been taken to be powers of the variable $\theta^3 = z$, namely z^N $(N = 1, 2, \ldots)$. Here, to allow for greater generality of interpretation, we leave $\lambda_N(z)$, $\mu_N(z)$ unspecified except to say that $\lambda_N(z)$ are a set of linearly independent functions in the interval $z_1 \leqslant z \leqslant z_2$. A parallel statement holds for the functions $\mu_N(z)$. For example, they can be Legendre polynomials or trigonometric functions.

Using the notation and definitions of § 2, we have

$$\left.\begin{aligned}\rho_{\mathrm{R}} A^{\frac{1}{2}} = \rho a^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}}\, \mathrm{d}z = \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}}\, \mathrm{d}z,\\ \rho_{\mathrm{R}} A^{\frac{1}{2}} y^{N0} = \rho a^{\frac{1}{2}} y^{N0} &= \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}} \lambda_N(z)\, \mathrm{d}z = \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} \lambda_N(z)\, \mathrm{d}z,\\ \rho_{\mathrm{R}} A^{\frac{1}{2}} y^{NM} = \rho a^{\frac{1}{2}} y^{NM} &= \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}} \lambda_N(z)\, \lambda_M(z)\, \mathrm{d}z = \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} \lambda_N(z)\, \lambda_M(z)\, \mathrm{d}z,\end{aligned}\right\} \tag{B 2}$$

$$\left.\begin{aligned}\boldsymbol{N}^{\alpha} a^{\frac{1}{2}} = {}_{\mathrm{R}}\boldsymbol{N}^{\alpha} A^{\frac{1}{2}} &= \int_{z_1}^{z_2} g^{\frac{1}{2}} \boldsymbol{t}^{\alpha}\, \mathrm{d}z = \int_{z_1}^{z_2} G^{\frac{1}{2}} \boldsymbol{t}_{\mathrm{R}}^{\alpha}\, \mathrm{d}z,\\ \boldsymbol{M}^{N\alpha} a^{\frac{1}{2}} = {}_{\mathrm{R}}\boldsymbol{M}^{N\alpha} A^{\frac{1}{2}} &= \int_{z_1}^{z_2} g^{\frac{1}{2}} \boldsymbol{t}^{\alpha} \lambda_N(z)\, \mathrm{d}z = \int_{z_1}^{z_2} G^{\frac{1}{2}} \boldsymbol{t}_{\mathrm{R}}^{\alpha} \lambda_N(z)\, \mathrm{d}z,\\ \boldsymbol{k}^{N} a^{\frac{1}{2}} = {}_{\mathrm{R}}\boldsymbol{k}^{N} A^{\frac{1}{2}} &= \int_{z_1}^{z_2} g^{\frac{1}{2}} \boldsymbol{t}^{3} \lambda_N'(z)\, \mathrm{d}z = \int_{z_1}^{z_2} G^{\frac{1}{2}} \boldsymbol{t}_{\mathrm{R}}^{3} \lambda_N'(z)\, \mathrm{d}z,\end{aligned}\right\} \tag{B 3}$$

$$\left.\begin{aligned}\rho \boldsymbol{f} a^{\frac{1}{2}} = \rho_{\mathrm{R}} \boldsymbol{f} A^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} \boldsymbol{f}^*\, \mathrm{d}z + [\boldsymbol{t}(gg^{33})^{\frac{1}{2}}]_{z=z_1} + [\boldsymbol{t}(gg^{33})^{\frac{1}{2}}]_{z=z_2}\\ &= \int_{z_1}^{z_2} \rho_{\mathrm{R}} G^{\frac{1}{2}} \boldsymbol{f}^*\, \mathrm{d}z + [\boldsymbol{t}_{\mathrm{R}}(GG^{33})^{\frac{1}{2}}]_{z=z_1} + [\boldsymbol{t}_{\mathrm{R}}(GG^{33})^{\frac{1}{2}}]_{z=z_2},\\ \rho \boldsymbol{f}_{\mathrm{e}} a^{\frac{1}{2}} = \rho_{\mathrm{R}} \boldsymbol{f}_{\mathrm{e}} A^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} \boldsymbol{f}_{\mathrm{e}}^*\, \mathrm{d}z = \int_{z_1}^{z_2} \rho_{\mathrm{R}} G^{\frac{1}{2}} \boldsymbol{f}_{\mathrm{e}}^*\, \mathrm{d}z,\\ \rho \boldsymbol{l}^{N} a^{\frac{1}{2}} = \rho_{\mathrm{R}} \boldsymbol{l}^{N} A^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} \boldsymbol{f}^* \lambda_N(z)\, \mathrm{d}z + [\boldsymbol{t}\lambda_N(z)\, (gg^{33})^{\frac{1}{2}}]_{z=z_1} + [\boldsymbol{t}\lambda_N(z)\, (gg^{33})^{\frac{1}{2}}]_{z=z_2},\\ &= \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}} \boldsymbol{f}^* \lambda_N(z)\, \mathrm{d}z + [\boldsymbol{t}_{\mathrm{R}} \lambda_N(z)\, (GG^{33})^{\frac{1}{2}}]_{z=z_1} + [\boldsymbol{t}_{\mathrm{R}} \lambda_N(z)\, (GG^{33})^{\frac{1}{2}}]_{z=z_2},\\ \rho \boldsymbol{l}_{\mathrm{e}}^{N} a^{\frac{1}{2}} = \rho_{\mathrm{R}} \boldsymbol{l}_{\mathrm{e}}^{N} A^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} \boldsymbol{f}_{\mathrm{e}}^* \lambda_N(z)\, \mathrm{d}z = \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}} \boldsymbol{f}_{\mathrm{e}}^* \lambda_N(z)\, \mathrm{d}z,\\ \rho \boldsymbol{c}_{\mathrm{e}} a^{\frac{1}{2}} = \rho_{\mathrm{R}} \boldsymbol{c}_{\mathrm{e}} A^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} \boldsymbol{c}_{\mathrm{e}}^*\, \mathrm{d}z = \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}} \boldsymbol{c}_{\mathrm{e}}^*\, \mathrm{d}z,\\ \rho \overline{w} a^{\frac{1}{2}} = \rho_{\mathrm{R}} \overline{w} A^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} (w_{\mathrm{e}}^* + \tfrac{1}{2} \boldsymbol{\Gamma}_{\mathrm{e}}^* \cdot \boldsymbol{L}^*)\, \mathrm{d}z = \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}} (w_{\mathrm{e}}^* + \tfrac{1}{2} \boldsymbol{\Gamma}_{\mathrm{e}}^* \cdot \boldsymbol{L}^*)\, \mathrm{d}z,\end{aligned}\right\} \tag{B.4}$$

$$\left.\begin{aligned}\rho \eta a^{\frac{1}{2}} = \rho_{\mathrm{R}} \eta A^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} \eta^*\, \mathrm{d}z = \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}} \eta^*\, \mathrm{d}z,\\ \rho \eta_N a^{\frac{1}{2}} = \rho_{\mathrm{R}} \eta_N A^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} \eta^* \mu_N(z)\, \mathrm{d}z = \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}} \eta^* \mu_N(z)\, \mathrm{d}z,\end{aligned}\right\} \tag{B 5}$$

$$\left.\begin{aligned}\rho s a^{\frac{1}{2}} = \rho_{\mathrm{R}} s A^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} s^*\, \mathrm{d}z - [k^* (gg^{33})^{\frac{1}{2}}]_{z=z_1} - [k^* (gg^{33})^{\frac{1}{2}}]_{z=z_2}\\ &= \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}} s^*\, \mathrm{d}z - [k_{\mathrm{R}}^* (GG^{33})^{\frac{1}{2}}]_{z=z_1} - [k_{\mathrm{R}}^* (GG^{33})^{\frac{1}{2}}]_{z=z_2},\\ \rho s_N a^{\frac{1}{2}} = \rho_{\mathrm{R}} s_N A^{\frac{1}{2}} &= \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}} s^* \mu_N(z)\, \mathrm{d}z - [k^* (gg^{33})^{\frac{1}{2}} \mu_N(z)]_{z=z_1} - [k^* (gg^{33})^{\frac{1}{2}} \mu_N(z)]_{z=z_2}\\ &= \int_{z_1}^{z_2} \rho_{\mathrm{R}}^* G^{\frac{1}{2}} s^* \mu_N(z)\, \mathrm{d}z - [k_{\mathrm{R}}^* (GG^{33})^{\frac{1}{2}} \mu_N(z)]_{z=z_1} - [k_{\mathrm{R}}^* (GG^{33})^{\frac{1}{2}} \mu_N(z)]_{z=z_2},\end{aligned}\right\} \tag{B 6}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$$\left.\begin{aligned}\rho\xi a^{\frac{1}{2}} &= \rho\xi A^{\frac{1}{2}} = \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}}\xi^* \,\mathrm{d}z = \int_{z_1}^{z_2} \rho^*_{\mathrm{R}} G^{\frac{1}{2}}\xi^* \,\mathrm{d}z,\\ \rho\xi_N a^{\frac{1}{2}} &= \rho_{\mathrm{R}}\xi_N A^{\frac{1}{2}} = \int_{z_1}^{z_2} \rho^* g^{\frac{1}{2}}\xi^* \mu_N(z)\,\mathrm{d}z + \int_{z_1}^{z_2} g^{\frac{1}{2}}\boldsymbol{p}^*\cdot\boldsymbol{g}^3\cdot\mu_N'(z)\,\mathrm{d}z\\ &= \int_{z_1}^{z_2} \rho^*_{\mathrm{R}} G^{\frac{1}{2}}\xi^*\mu_N(z)\,\mathrm{d}z + \int_{z_1}^{z_2} G^{\frac{1}{2}}\boldsymbol{p}^*_{\mathrm{R}}\cdot\boldsymbol{G}^3\cdot\mu_N'(z)\,\mathrm{d}z,\end{aligned}\right\} \tag{B 7}$$

$$\left.\begin{aligned}\boldsymbol{p}\cdot\boldsymbol{a}^\alpha a^{\frac{1}{2}} &= {}_{\mathrm{R}}\boldsymbol{p}\cdot\boldsymbol{A}^\alpha A^{\frac{1}{2}} = \int_{z_1}^{z_2} g^{\frac{1}{2}}\boldsymbol{p}^*\cdot\boldsymbol{g}^\alpha\,\mathrm{d}z = \int_{z_1}^{z_2} G^{\frac{1}{2}}\boldsymbol{p}^*_{\mathrm{R}}\cdot\boldsymbol{G}^\alpha\,\mathrm{d}z,\\ \boldsymbol{p}_N\cdot\boldsymbol{a}^\alpha a^{\frac{1}{2}} &= {}_{\mathrm{R}}\boldsymbol{p}_N\cdot\boldsymbol{A}^\alpha A^{\frac{1}{2}} = \int_{z_1}^{z_2} g^{\frac{1}{2}}\boldsymbol{p}^*\cdot\boldsymbol{g}^\alpha\cdot\mu_N(z)\,\mathrm{d}z = \int_{z_1}^{z_2} G^{\frac{1}{2}}\boldsymbol{p}^*_{\mathrm{R}}\cdot\boldsymbol{G}^\alpha\cdot\mu_N(z)\,\mathrm{d}z.\end{aligned}\right\} \tag{B 8}$$

In (B 3)–(B 7), $\lambda_N'(z) = \mathrm{d}\lambda_N(z)/\mathrm{d}z$, $\mu_N'(z) = \mathrm{d}\mu_N(z)/\mathrm{d}z$.

We turn now to the development of a theory for the electromagnetic fields based on Cosserat surfaces. It is convenient to introduce two other sets of functions $\psi_N(z)$, $\chi_N(z)$ in the interval $z_1 \leqslant z \leqslant z_2$, this time for $N = 0, 1, \ldots, L$, with the properties

$$\int_{z_1}^{z_2} \psi_N(z)\,\psi_M(z)\,\mathrm{d}z = \delta_{MN}, \quad \int_{z_1}^{z_2} \chi_N(z)\,\chi_M(z)\,\mathrm{d}z = \delta_{MN}, \tag{B 9}$$

$$\frac{\mathrm{d}\psi_N}{\mathrm{d}z} = \sum_{K=0}^{N} \psi_N^K \chi_K(z), \quad \frac{\mathrm{d}\chi_N}{\mathrm{d}z} = \sum_{K=0}^{N} \chi_N^K \psi_K(z). \tag{B 10}$$

We mention two examples. If $P_N(\mu)$ is a Legendre polynomial of order N we set

$$\left.\begin{aligned}\chi_N(z) = \psi_N(z) = \left(\frac{2N+1}{z_2-z_1}\right)^{\frac{1}{2}} P_N(\mu), \quad \mu = \frac{2z-z_1-z_2}{z_2-z_1} \quad (N = 0, 1, 2, \ldots),\\ \chi_N^K = \psi_N^K = c_N^K,\end{aligned}\right\} \tag{B 11}$$

where

$$\left.\begin{aligned}c_N^K &= 2(2N+1)^{\frac{1}{2}}(2K+1)^{\frac{1}{2}}/(z_2-z_1), \quad K = 0, 2, \ldots, N-1; \quad c_N^K = 0,\\ &\qquad K = 1, 3, \ldots, (N \text{ odd})\\ c_N^K &= 2(2N+1)^{\frac{1}{2}}(2K+1)^{\frac{1}{2}}/(z_2-z_1), \quad K = 1, 3, \ldots, N-1; \quad c_N^K = 0,\\ &\qquad K = 0, 2, \ldots, (N \text{ even or } N = 0).\end{aligned}\right\} \tag{B 12}$$

The second example uses orthogonal trigonometric functions of the form

$$\left.\begin{aligned}\psi_N &= 2^{\frac{1}{2}}(z_2-z_1)^{-\frac{1}{2}}\sin\{\tfrac{1}{2}N\pi(1+\mu)\},\\ \psi_N^K &= 0 \quad (K \neq N), \quad \psi_N^K = N\pi(z_2-z_1)^{-1} \quad (K = N),\\ \chi_N &= (z_2-z_1)^{-\frac{1}{2}} \quad (N = 0), \quad \chi_N = 2^{\frac{1}{2}}(z_2-z_1)^{-\frac{1}{2}}\cos\{\tfrac{1}{2}N\pi(1+\mu)\} \quad (N = 1, 2, \ldots),\\ \chi_N^K &= 0 \quad (K \neq N), \quad \chi_N^K = -N\pi(z_2-z_1)^{-1} \quad (K = N),\end{aligned}\right\} \tag{B 13}$$

or we may interchange the functions ψ_N and χ_N, ψ_N^K and χ_N^K.

We first deal with the spatial forms (A 1)–(A 3) of the electromagnetic equations and the corresponding field equations (A 10). Multiplying $(\mathrm{A}\ 10)_{2,4}$ by $\chi_N(z)$ znd $\psi_N(z)$, respectively, and integrating over a material region $\mathscr{P}^*$ in the configuration at time t gives

$$\int_{\partial\mathscr{P}^*} \chi_N(z)\,\boldsymbol{b}\cdot\mathrm{d}\boldsymbol{a} = \int_{\mathscr{P}^*} \chi_N'(z)\,\boldsymbol{g}^3\cdot\boldsymbol{b}\,\mathrm{d}v, \tag{B 14}$$

$$\int_{\partial\mathscr{P}^*} \psi_N(z)\,\bar{\boldsymbol{d}}\cdot\mathrm{d}\boldsymbol{a} = \int_{\mathscr{P}^*} (\psi_N(z)\,e + \psi_N'(z)\,\boldsymbol{g}^3\cdot\bar{\boldsymbol{d}})\,\mathrm{d}v, \tag{B 15}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

where a prime denotes derivative with respect to z. Next, we take the scalar product of $(\text{A }10)_1$ with $\boldsymbol{g}^3\psi_N(z)$ and integrate over $\mathscr{P}^*$; also we take the linear transformation of $(\text{A }10)_1$ with $\chi_N(z)\,\boldsymbol{a}_\alpha\otimes\boldsymbol{g}^\alpha$ and integrate over $\mathscr{P}^*$. This yields the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\psi_N(z)\,\boldsymbol{b}\cdot\boldsymbol{g}^3\,\mathrm{d}v=\int_{\partial\mathscr{P}^*}\psi_N(z)\,\boldsymbol{g}^3\times\boldsymbol{e}^*\cdot\mathrm{d}\boldsymbol{a}, \tag{B 16a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\chi_N(z)\,b^\alpha\boldsymbol{a}_\alpha\,\mathrm{d}v=\int_{\mathscr{P}^*}\{\chi_N(z)\,b^\alpha\boldsymbol{L}\boldsymbol{a}_\alpha+\chi_N'(z)\,\boldsymbol{a}_\alpha[\boldsymbol{g}^\alpha\boldsymbol{g}^3\boldsymbol{e}^*]\}\,\mathrm{d}v+\int_{\partial\mathscr{P}^*}\chi_N(z)\,\boldsymbol{a}_\alpha[\boldsymbol{g}^\alpha\times\boldsymbol{e}^*]\cdot\mathrm{d}\boldsymbol{a}. \tag{B 16b}$$

Similarly, from $(\text{A }10)_3$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\chi_N(z)\,\bar{\boldsymbol{d}}\cdot\boldsymbol{g}^3\,\mathrm{d}v=-\int_{\partial\mathscr{P}^*}\chi_N(z)\,\boldsymbol{g}^3\times\boldsymbol{h}^*\cdot\mathrm{d}\boldsymbol{a}-\int_{\mathscr{P}^*}\chi_N(z)\,\boldsymbol{j}^*\cdot\boldsymbol{g}^3\,\mathrm{d}v, \tag{B 17a}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}^*}\psi_N(z)\,\bar{\boldsymbol{d}}^\alpha\boldsymbol{a}_\alpha\,\mathrm{d}v=\int_{\mathscr{P}^*}\{&-\psi_N(z)\,\bar{\boldsymbol{d}}^\alpha\boldsymbol{L}\boldsymbol{a}_\alpha+\psi_N'(z)\,\boldsymbol{a}_\alpha[\boldsymbol{g}^\alpha\boldsymbol{g}^3\boldsymbol{h}^*]\\&+\boldsymbol{a}_\alpha(\boldsymbol{j}^*\cdot\boldsymbol{g}^\alpha)\,\psi_N(z)\}\,\mathrm{d}v+\int_{\partial\mathscr{P}^*}\psi_N(z)\,\boldsymbol{a}_\alpha[\boldsymbol{g}^\alpha\times\boldsymbol{h}^*]\cdot\mathrm{d}\boldsymbol{a}.\end{aligned} \tag{B 17b}$$

In (B 16) and (B 17) we have used the tensor $\boldsymbol{L}$ where

$$\left.\begin{aligned}\boldsymbol{L}=\dot{\boldsymbol{a}}_i\otimes\boldsymbol{a}^i,\quad \dot{\boldsymbol{a}}_i=\boldsymbol{L}\boldsymbol{a}_i,\quad \boldsymbol{a}^3=\boldsymbol{a}_3,\\ \dot{\boldsymbol{F}}=\boldsymbol{L}\boldsymbol{F},\quad \boldsymbol{F}=\boldsymbol{a}_i\otimes\boldsymbol{A}^i,\quad \boldsymbol{a}_i=\boldsymbol{F}\boldsymbol{A}_i,\quad \det\boldsymbol{F}=\Gamma=a^{\frac{1}{2}}/A^{\frac{1}{2}}.\end{aligned}\right\} \tag{B 18}$$

Similar equations in material form can be obtained from (A 21) if we replace $\mathscr{P}^*$, $\partial\mathscr{P}^*$, $\mathrm{d}\boldsymbol{a}$, $\mathrm{d}v$, $\boldsymbol{e}^*$, $\boldsymbol{h}^*$, $\bar{\boldsymbol{d}}$, $\boldsymbol{b}$, e, $\boldsymbol{j}^*$, $\boldsymbol{a}_\alpha$, $\boldsymbol{a}^\alpha$, $\boldsymbol{a}_3$ by $\mathscr{P}^*_{\mathrm{R}}$, $\partial\mathscr{P}^*_{\mathrm{R}}$, $\mathrm{d}\boldsymbol{A}$, $\mathrm{d}V$, $\boldsymbol{E}$, $\boldsymbol{H}$, $\bar{\boldsymbol{D}}$, $\boldsymbol{B}$, E, $\boldsymbol{J}$, $\boldsymbol{A}_\alpha$, $\boldsymbol{A}^\alpha$, $\boldsymbol{A}_3$, respectively, and omit all terms containing the tensor $\boldsymbol{L}$.

Equations (B 14)–(B 18), together with their material counterparts, are now applied to a shell-like body bounded by the surfaces $z=z_1$, $z=z_2$ and a surface $f(\theta^1,\theta^2)=0$. The resulting integrals are over material surfaces $\mathscr{P}$ in the surface $z=0$ in the present configuration at time t, bounded by $\partial\mathscr{P}$ whose outward unit normal in the surface is $\boldsymbol{\nu}$, with corresponding surfaces $\mathscr{P}_{\mathrm{R}}$ bounded by $\partial\mathscr{P}_{\mathrm{R}}$ in the reference configuration with outward unit normal ${}_{\mathrm{R}}\boldsymbol{\nu}$, where

$$\boldsymbol{\nu}=\nu_\alpha\boldsymbol{a}^\alpha=\nu^\alpha\boldsymbol{a}_\alpha,\quad {}_{\mathrm{R}}\boldsymbol{\nu}={}_{\mathrm{R}}\nu_\alpha\boldsymbol{A}^\alpha={}_{\mathrm{R}}\nu^\alpha\boldsymbol{A}_\alpha, \tag{B 19}$$

$$\int_{\partial\mathscr{P}}\boldsymbol{b}_N\cdot\boldsymbol{\nu}\,\mathrm{d}s=\int_{\mathscr{P}}\left(\sum_{K=0}^{N}\chi_N^K\boldsymbol{b}_K-\hat{\boldsymbol{b}}_N\right)\cdot\mathrm{d}\boldsymbol{\sigma}, \tag{B 20}$$

$$\int_{\partial\mathscr{P}}\bar{\boldsymbol{d}}_N\cdot\boldsymbol{\nu}\,\mathrm{d}s=\int_{\mathscr{P}}e_N\,\mathrm{d}\sigma+\int_{\mathscr{P}}\left(\sum_{K=0}^{N}\psi_N^K\bar{\boldsymbol{d}}_K-\hat{\boldsymbol{d}}_N\right)\cdot\mathrm{d}\boldsymbol{\sigma}, \tag{B 21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\boldsymbol{b}_N\cdot\mathrm{d}\boldsymbol{\sigma}=-\int_{\partial\mathscr{P}}\boldsymbol{e}_N^*\cdot\mathrm{d}\boldsymbol{r}, \tag{B 22a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}(\boldsymbol{a}_3\times\boldsymbol{b}_N)\times\mathrm{d}\boldsymbol{\sigma}=\int_{\partial\mathscr{P}}(\boldsymbol{e}_N^*\cdot\boldsymbol{a}_3)\,\mathrm{d}\boldsymbol{r}+\int_{\mathscr{P}}\boldsymbol{L}[(\boldsymbol{a}_3\times\boldsymbol{b}_N)\times\mathrm{d}\boldsymbol{\sigma}]+\int_{\mathscr{P}}\left(\hat{\boldsymbol{e}}_N^*-\sum_{K=0}^{N}\chi_N^K\boldsymbol{e}_K^*\right)\times\mathrm{d}\boldsymbol{\sigma}, \tag{B 22b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}\bar{\boldsymbol{d}}_N\cdot\mathrm{d}\boldsymbol{\sigma}=\int_{\partial\mathscr{P}}\boldsymbol{h}_N^*\cdot\mathrm{d}\boldsymbol{r}-\int_{\mathscr{P}}\boldsymbol{j}_N^*\cdot\mathrm{d}\boldsymbol{\sigma}, \tag{B 23a}$$

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{P}}(\boldsymbol{a}_3\times\bar{\boldsymbol{d}}_N)\times\mathrm{d}\boldsymbol{\sigma}=&\int_{\partial\mathscr{P}}(\boldsymbol{h}_N^*\cdot\boldsymbol{a}_3)\,\mathrm{d}\boldsymbol{r}+\int_{\mathscr{P}}\{(\boldsymbol{a}_3\times\boldsymbol{j}_N^*)\times\mathrm{d}\boldsymbol{\sigma}-\boldsymbol{L}[(\boldsymbol{a}_3\times\bar{\boldsymbol{d}}_N)\times\mathrm{d}\boldsymbol{\sigma}]\}\\&+\int_{\mathscr{P}}\left(\hat{\boldsymbol{h}}_N^*-\sum_{K=0}^{N}\psi_N^K\boldsymbol{h}_K^*\right)\times\mathrm{d}\boldsymbol{\sigma}.\end{aligned} \tag{B 23b}$$

Similar balance equations in material forms may be obtained by replacing $\boldsymbol{e}^*_N$, $\boldsymbol{h}^*_N$, $\boldsymbol{b}_N$, $\bar{\boldsymbol{d}}_N$, $\hat{\boldsymbol{e}}^*_N$, $\hat{\boldsymbol{h}}^*_N$, $\hat{\boldsymbol{b}}_N$, $\hat{\boldsymbol{d}}_N$, e_N, $\boldsymbol{j}^*_N$, $\boldsymbol{v}$, $\mathrm{d}\boldsymbol{\sigma}$, $\mathrm{d}\boldsymbol{r}$, $\boldsymbol{a}_i$, $\boldsymbol{a}^i$ by $\boldsymbol{E}_N$, $\boldsymbol{H}_N$, $\boldsymbol{B}_N$, $\bar{\boldsymbol{D}}_N$, $\hat{\boldsymbol{E}}_N$, $\hat{\boldsymbol{H}}_N$, $\hat{\boldsymbol{B}}_N$, $\hat{\boldsymbol{D}}_N$, E_N, $\boldsymbol{J}_N$, ${}_{\mathrm{R}}\boldsymbol{v}$, $\mathrm{d}\boldsymbol{\sigma}_{\mathrm{R}}$, $\mathrm{d}\boldsymbol{R}$, $\boldsymbol{A}_i$, $\boldsymbol{A}^i$, respectively, and omitting the terms containing the tensor $\boldsymbol{L}$. In these balances we have used the following definitions:

$$\left.\begin{aligned}
&\mathrm{d}\boldsymbol{\sigma} = \boldsymbol{a}_3\,\mathrm{d}\sigma, \quad \mathrm{d}\sigma_{\mathrm{R}} = \boldsymbol{A}_3\,\mathrm{d}\sigma_{\mathrm{R}},\\
&\boldsymbol{e}^*_N = e^*_{Ni}\boldsymbol{a}^i, \quad \boldsymbol{h}^*_N = h^*_{Ni}\boldsymbol{a}^i, \quad \boldsymbol{b}_N = b^i_N\boldsymbol{a}_i, \quad \bar{\boldsymbol{d}}_N = \bar{d}^i_N\boldsymbol{a}_i,\\
&\boldsymbol{j}^*_N = j^{*i}_N\boldsymbol{a}_i, \quad \hat{\boldsymbol{e}}^*_N = [\chi_N(z)\,e^*_i\,\boldsymbol{a}^i]^{z_2}_{z_1}, \quad \hat{\boldsymbol{h}}^*_N = [\psi_N(z)\,h^*_i\boldsymbol{a}^i]^{z_2}_{z_1},\\
&\hat{\boldsymbol{b}}_N = [\chi_N(z)\,b^i\boldsymbol{a}_i g^{\frac{1}{2}}/a^{\frac{1}{2}}]^{z_2}_{z_1}, \quad \hat{\boldsymbol{d}}_N = [\psi_N(z)\,\bar{d}^i\boldsymbol{a}_i\,g^{\frac{1}{2}}/a^{\frac{1}{2}}]^{z_2}_{z_1},\\
&\boldsymbol{E}_N = E_{Ni}\boldsymbol{A}^i, \quad \boldsymbol{H}_N = H_{Ni}\boldsymbol{A}^i, \quad \boldsymbol{B}_N = B^i_N\boldsymbol{A}_i, \quad \bar{\boldsymbol{D}}_N = \bar{D}^i_N\boldsymbol{A}_i,\\
&\boldsymbol{J}_N = J^i_N\boldsymbol{A}_i, \quad \hat{\boldsymbol{E}}_N = [\chi_N(z)\,E_i\,\boldsymbol{A}^i]^{z_2}_{z_1}, \quad \hat{\boldsymbol{H}}_N = [\psi_N(z)\,H_i\,\boldsymbol{A}^i]^{z_2}_{z_1},\\
&\hat{\boldsymbol{B}}_N = [\chi_N(z)\,B^i\boldsymbol{A}_i G^{\frac{1}{2}}/A^{\frac{1}{2}}]^{z_2}_{z_1}, \quad \hat{\boldsymbol{D}}_N = [\psi_N(z)\,\bar{D}^i\boldsymbol{A}_i G^{\frac{1}{2}}/A^{\frac{1}{2}}]^{z_2}_{z_1},
\end{aligned}\right\} \tag{B 24}$$

$$\boldsymbol{E}_N = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{e}^*_N, \quad \boldsymbol{H}_N = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{h}^*_N, \quad \Gamma\bar{\boldsymbol{d}}_N = \boldsymbol{F}\bar{\boldsymbol{D}}_N, \quad \Gamma\boldsymbol{b}_N = \boldsymbol{F}\boldsymbol{B}_N, \quad \Gamma\boldsymbol{j}^*_N = \boldsymbol{F}\boldsymbol{J}_N, \quad \Gamma e_N = E_N, \tag{B 25}$$

where $\boldsymbol{F}$, Γ are given in (B 18) and where, recalling (A 25),

$$A^{\frac{1}{2}}B^\alpha_N = a^{\frac{1}{2}}b^\alpha_N = \int_{z_1}^{z_2}\chi_N(z)\,G^{\frac{1}{2}}B^\alpha\,\mathrm{d}z, \quad A^{\frac{1}{2}}B^3_N = a^{\frac{1}{2}}b^3_N = \int_{z_1}^{z_2}\psi_N(z)\,G^{\frac{1}{2}}B^3\,\mathrm{d}z, \tag{B 26a, b}$$

$$A^{\frac{1}{2}}\bar{D}^\alpha_N = a^{\frac{1}{2}}\bar{d}^\alpha_N = \int_{z_1}^{z_2}\psi_N(z)\,G^{\frac{1}{2}}\bar{D}^\alpha\,\mathrm{d}z, \quad A^{\frac{1}{2}}\bar{D}^3_N = a^{\frac{1}{2}}\bar{d}^3_N = \int_{z_1}^{z_2}\chi_N(z)\,G^{\frac{1}{2}}\bar{D}^3\,\mathrm{d}z, \tag{B 27a, b}$$

$$E_{N\alpha} = e^*_{N\alpha} = \int_{z_1}^{z_2}\psi_N(z)\,E_\alpha\,\mathrm{d}z, \quad E_{N3} = e^*_{N3} = \int_{z_1}^{z_2}\chi_N(z)\,E_3\,\mathrm{d}z, \tag{B 28a, b}$$

$$H_{N\alpha} = h^*_{N\alpha} = \int_{z_1}^{z_2}\chi_N(z)\,H_\alpha\,\mathrm{d}z, \quad H_{N3} = h^*_{N3} = \int_{z_1}^{z_2}\psi_N(z)\,H_3\,\mathrm{d}z, \tag{B 29a, b}$$

$$A^{\frac{1}{2}}J^\alpha_N = a^{\frac{1}{2}}j^{*\alpha}_N = \int_{z_1}^{z_2}\psi_N(z)\,G^{\frac{1}{2}}J^\alpha\,\mathrm{d}z, \quad A^{\frac{1}{2}}J^3_N = a^{\frac{1}{2}}j^{*3}_N = \int_{z_1}^{z_2}\chi_N(z)\,G^{\frac{1}{2}}J^3\,\mathrm{d}z, \tag{B 30a, b}$$

$$A^{\frac{1}{2}}E_N = a^{\frac{1}{2}}e_N = \int_{z_1}^{z_2}\psi_N(z)\,G^{\frac{1}{2}}E\,\mathrm{d}z. \tag{B 31}$$

Finally, in this Appendix, we record some results that arise when the position vector, velocity vector and electromagnetic vectors $\boldsymbol{e}^*$, $\boldsymbol{h}^*$, $\boldsymbol{E}$, $\boldsymbol{H}$ in the shell have the approximate representations

$$\left.\begin{aligned}
&\boldsymbol{r}^* = \boldsymbol{r} + \sum_{N=1}^{P}\lambda_N(z)\,\boldsymbol{d}_N, \quad \boldsymbol{v}^* = \boldsymbol{v} + \sum_{N=1}^{P}\lambda_N(z)\,\boldsymbol{w}_N,\\
&\boldsymbol{e}^* = \sum_{N=0}^{L}\{\psi_N(z)\,e^*_{N\alpha}\,\boldsymbol{g}^\alpha + \chi_N(z)\,e^*_{N3}\,\boldsymbol{g}^3\}, \quad \boldsymbol{h}^* = \sum_{N=0}^{L}\{\chi_N(z)\,h^*_{N\alpha}\,\boldsymbol{g}^\alpha + \psi_N(z)\,h^*_{N3}\,\boldsymbol{g}^3\},\\
&\boldsymbol{E} = \sum_{N=0}^{L}\{\psi_N(z)\,E_{N\alpha}\,\boldsymbol{G}^\alpha + \chi_N(z)\,E_{N3}\,\boldsymbol{G}^3\}, \quad \boldsymbol{H} = \sum_{N=0}^{L}\{\chi_N(z)\,H_{N\alpha}\,\boldsymbol{G}^\alpha + \psi_N(z)\,H_{N3}\,\boldsymbol{G}^3\},
\end{aligned}\right\} \tag{B 32}$$

where, in view of the orthonormal properties of $\psi_N(z)$, $\chi_N(z)$ in the interval $z_1 \leqslant z \leqslant z_2$, e^*_{Ni}, h^*_{Ni}, E_{Ni}, H_{Ni} are given by (B 28a)–(B 29b). From (A 26), (B 32) and (B 26)–(B 31), it follows that

$$\begin{aligned}
\int_{z_1}^{z_2}\rho^* g^{\frac{1}{2}}(w^*_{\mathrm{e}} + \tfrac{1}{2}\boldsymbol{\Gamma}^*_{\mathrm{e}}\cdot\boldsymbol{L}^*)\,\mathrm{d}z &= a^{\frac{1}{2}}\{\boldsymbol{N}^\alpha_{\mathrm{e}}\cdot\boldsymbol{v}_{,\alpha} + \sum_{N=1}^{P}(\boldsymbol{k}^N_{\mathrm{e}}\cdot\boldsymbol{w}_N + \boldsymbol{M}^{N\alpha}_{\mathrm{e}}\cdot\boldsymbol{w}_{N,\alpha})\\
&\quad + \sum_{N=0}^{L}\{\boldsymbol{e}^*_N\cdot\boldsymbol{j}^*_N + \boldsymbol{e}^*_N\cdot(\dot{\bar{\boldsymbol{d}}}_N + \bar{\boldsymbol{d}}_N\,\mathrm{div}_s\,\boldsymbol{v} - \boldsymbol{L}\bar{\boldsymbol{d}}_N) + \boldsymbol{h}^*_N\cdot(\dot{\boldsymbol{b}}_N + \boldsymbol{b}_N\,\mathrm{div}_s\,\boldsymbol{v} - \boldsymbol{L}\boldsymbol{b}_N)\},
\end{aligned} \tag{B 33}$$

$$\begin{aligned}
\int_{z_1}^{z_2}\rho^*_{\mathrm{R}} G^{\frac{1}{2}}(w^*_{\mathrm{e}} + \tfrac{1}{2}\boldsymbol{\Gamma}^*_{\mathrm{e}}\cdot\boldsymbol{L}^*)\,\mathrm{d}z &= A^{\frac{1}{2}}\{{}_{\mathrm{R}}\boldsymbol{N}^\alpha_{\mathrm{e}}\cdot\boldsymbol{v}_{,\alpha} + \sum_{N=1}^{P}({}_{\mathrm{R}}\boldsymbol{k}^N_{\mathrm{e}}\cdot\boldsymbol{w}_N + {}_{\mathrm{R}}\boldsymbol{M}^{N\alpha}_{\mathrm{e}}\cdot\boldsymbol{w}_{N,\alpha})\\
&\quad + \sum_{N=0}^{L}(\boldsymbol{E}_N\cdot\boldsymbol{J}_N + \boldsymbol{E}_N\cdot\dot{\bar{\boldsymbol{D}}}_N + \boldsymbol{H}_N\cdot\dot{\boldsymbol{B}}_N)\},
\end{aligned} \tag{B 34}$$

where $\boldsymbol{N}^{\alpha}_{\mathrm{e}}$, ${}_{\mathrm{R}}\boldsymbol{N}^{\alpha}_{\mathrm{e}}$, $\boldsymbol{k}_{\mathrm{e}}$, ${}_{\mathrm{R}}\boldsymbol{k}_{\mathrm{e}}$, $\boldsymbol{M}^{\alpha}_{\mathrm{e}}$, ${}_{\mathrm{R}}\boldsymbol{M}^{\alpha}_{\mathrm{e}}$ are given in terms of $\boldsymbol{t}^{i}_{\mathrm{e}}$, ${}_{\mathrm{R}}\boldsymbol{t}^{i}_{\mathrm{e}}$ by formulae of the same type as (B 3). Also

$$\int_{z_1}^{z_2} \rho \boldsymbol{c}^{*}_{\mathrm{e}}\, g^{\frac{1}{2}}\,\mathrm{d}z = a^{\frac{1}{2}}\left\{\boldsymbol{a}_{\alpha}\times \boldsymbol{N}^{\alpha}_{\mathrm{e}} + \sum_{N=1}^{P}\left(\boldsymbol{d}_{N}\times \boldsymbol{k}^{N}_{\mathrm{e}} + \boldsymbol{d}_{N,\alpha}\times \boldsymbol{M}^{N\alpha}_{\mathrm{e}}\right)\right\}$$

$$= A^{\frac{1}{2}}\left\{\boldsymbol{a}_{\alpha}\times {}_{\mathrm{R}}\boldsymbol{N}^{\alpha}_{\mathrm{e}} + \sum_{N=1}^{P}\left(\boldsymbol{d}_{N}\times {}_{\mathrm{R}}\boldsymbol{k}^{N}_{\mathrm{e}} + \boldsymbol{d}_{N,\alpha}\times {}_{\mathrm{R}}\boldsymbol{M}^{N\alpha}_{\mathrm{e}}\right)\right\}. \qquad \text{(B 35)}$$

Appendix C

This Appendix contains a brief discussion of the specific Helmholtz free energy response function ψ^* in the three-dimensional theory of magnetic, polarized thermoelastic shells, along with related details arising from symmetry restrictions. In its reference configuration, the shell is assumed homogeneous and is of constant reference density and constant reference temperature. The Helmholtz free energy function ψ^* for such a medium has the form

$$\psi^* = \psi^*(g_{ij}, \theta^*, E_i, H_i; G_{ij}), \qquad \text{(C 1)}$$

where the dependence on the constant reference temperature Θ has not been displayed. We suppose that the position vector of the reference configuration of the body is specified by

$$\boldsymbol{R}^* = \boldsymbol{R} + \sum_{N=1}^{\infty} \lambda_N(z)\,\boldsymbol{D}_N \qquad \text{(C 2)}$$

and that the position vector and temperature of the body in the configuration at time t are given by

$$\boldsymbol{r}^* = \boldsymbol{r} + \sum_{N=1}^{\infty} \lambda_N(z)\,\boldsymbol{d}_N, \quad \theta^* = \theta + \sum_{M=1}^{\infty} \mu_M(z)\,\theta_M, \quad \boldsymbol{d}_N = d_{Ni}\,\boldsymbol{a}^i, \quad \boldsymbol{d}_{N,\alpha} = d_{Ni\alpha}\,\boldsymbol{a}^i. \qquad \text{(C 3)}$$

Also the components of the electric and magnetic vectors at time t are represented by

$$E_\alpha = \sum_{N=0}^{\infty} \psi_N(z)\,E_{N\alpha}, \quad H_\alpha = \sum_{N=0}^{\infty} \chi_N(z)\,H_{N\alpha}, \quad E_3 = \sum_{N=0}^{\infty} \chi_N(z)\,E_{N3}, \quad H_3 = \sum_{N=0}^{\infty} \psi_N(z)\,H_{N3}. \qquad \text{(C 4)}$$

Then (C 1) may be expressed in the form

$$\psi^* = \hat{\psi}^*(a_{\alpha\beta}, d_{\mathrm{R}i}, d_{\mathrm{R}i\alpha}, \theta, \theta_S, E_{Mi}, H_{Mi}; D_{\mathrm{R}i}, D_{\mathrm{R}i\alpha}, A_{\alpha\beta}, z). \qquad \text{(C 5)}$$

We consider now two cases in which the functions chosen for λ_N, μ_M, ψ_N, χ_N have different properties when z is changed to $-z$.

case (a)

$$\left.\begin{aligned} \lambda_N(-z) &= (-1)^N \lambda_N(z), \quad \mu_M(-z) = (-1)^M \mu_M(z),\\ \psi_N(-z) &= (-1)^N \psi_N(z), \quad \chi_M(-z) = (-1)^M \chi_M(z). \end{aligned}\right\} \qquad \text{(C 6)}$$

In view of (C 6), ψ^* in (C 5) has the property

$$\begin{aligned} &\hat{\psi}^*(a_{\alpha\beta}, d_{\mathrm{R}i}, d_{\mathrm{R}i\alpha}, \theta, \theta_S, E_{Mi}, H_{Mi}; D_{\mathrm{R}i}, D_{\mathrm{R}i\alpha}, A_{\alpha\beta}, -z)\\ &\quad = \hat{\psi}^*(a_{\alpha\beta}, (-1)^{R+1} d_{\mathrm{R}i}, (-1)^R d_{\mathrm{R}i\alpha}, \theta, (-1)^S \theta_S, (-1)^M E_{Mi}, (-1)^M H_{Mi};\\ &\qquad (-1)^{R+1} D_{\mathrm{R}i}, (-1)^R D_{\mathrm{R}i\alpha}, A_{\alpha\beta}, z). \end{aligned} \qquad \text{(C 7)}$$

case (b)

$$\left.\begin{aligned} \lambda_N(-z) &= (-1)^N \lambda_N(z), \quad \mu_M(-z) = (-1)^M \mu_M(z),\\ \psi_N(-z) &= (-1)^{N+1} \psi_N(z), \quad \chi_M(-z) = (-1)^M \chi_M(z), \end{aligned}\right\} \qquad \text{(C 8)}$$

which implies that

$$\hat{\psi}^*(a_{\alpha\beta}, d_{\mathrm{R}i}, d_{\mathrm{R}i\alpha}, \theta, \theta_S, E_{M\alpha}, E_{M3}, H_{M\alpha}, H_{M3}, D_{\mathrm{R}i}, D_{\mathrm{R}i\alpha}, A_{\alpha\beta}, -z)$$
$$= \hat{\psi}^*(a_{\alpha\beta}, (-1)^{R+1} d_{\mathrm{R}i}, (-1)^R d_{\mathrm{R}i\alpha}, \theta, (-1)^S \theta_S, (-1)^{M+1} E_{M\alpha}, (-1)^M E_{M3},$$
$$(-1)^M H_{M\alpha}, (-1)^{M+1} H_{M3}, (-1)^{R+1} D_{\mathrm{R}i}, (-1)^R D_{\mathrm{R}i\alpha}, A_{\alpha\beta}, z). \quad \text{(C 9)}$$

Given the expansions (C 2)–(C 4), the complete theory may be represented in a form similar to the results in the theory of Cosserat surfaces with an infinity of directors, along with temperatures and electromagnetic variables. The two-dimensional Helmholtz free energy function ψ that corresponds to (C 5) is then given by

$$\rho a^{\frac{1}{2}}\psi = \rho_{\mathrm{R}} A^{\frac{1}{2}}\psi = \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \rho_{\mathrm{R}}^* G^{\frac{1}{2}}\psi^* \,\mathrm{d}z \quad \text{(C 10)}$$

provided the material surface $\mathscr{S}$ of the Cosserat surfaces $\mathscr{C}_K$ is identified with the middle surfaces of the shell of thickness h in the reference configuration. Then, with this geometrical symmetry, since ρ_{R}^* is constant and $G^{\frac{1}{2}}$ has symmetry properties similar to those of ψ^*, it follows from (C 7) and (C 9) that ψ satisfies the conditions

case (*a*)

$$\psi(a_{\alpha\beta}, (-1)^{R+1} d_{\mathrm{R}i}, (-1)^R d_{\mathrm{R}i\alpha}, \theta, (-1)^S \theta_S, (-1)^M E_{Mi},$$
$$(-1)^M H_{Mi,\,i}, (-1)^{R+1} D_{\mathrm{R}i}, (-1)^R D_{\mathrm{R}i\alpha}, A_{\alpha\beta})$$
$$= \psi(a_{\alpha\beta}, d_{\mathrm{R}i}, d_{\mathrm{R}i\alpha}, \theta, \theta_S, E_{Mi}, H_{Mi}, D_{\mathrm{R}i}, D_{\mathrm{R}i\alpha}, A_{\alpha\beta}), \quad \text{(C 11)}$$

case (*b*)

$$\psi(a_{\alpha\beta}, (-1)^{R+1} d_{\mathrm{R}i}, (-1)^R d_{\mathrm{R}i\alpha}, \theta, (-1)^S \theta_S, (-1)^{M+1} E_{M\alpha}, (-1)^M E_{M3}, (-1)^M H_{M\alpha},$$
$$(-1)^{M+1} E_{M3}, (-1)^{R+1} D_{\mathrm{R}i}, (-1)^R D_{\mathrm{R}i\alpha}, A_{\alpha\beta})$$
$$= \psi(a_{\alpha\beta}, d_{\mathrm{R}i}, d_{\mathrm{R}i\alpha}, \theta, \theta_S, \theta_s, E_{M\alpha}, E_{M3}, H_{M\alpha}, H_{M3}, D_{\mathrm{R}i}, D_{\mathrm{R}i\alpha}, A_{\alpha\beta}). \quad \text{(C 12)}$$

Next, suppose that the material of the shell has symmetry with respect to reflexions along the normal direction $\boldsymbol{A}_3$ to its reference surface. Then, making use of (C 5) and (C 7), we have

case (*a*)

$$\hat{\psi}^*(a_{\alpha\beta}, d_{\mathrm{R}\alpha}, d_{\mathrm{R}3}, d_{\mathrm{R}\alpha\beta}, d_{\mathrm{R}3\alpha}, \theta, \theta_S, E_{M\alpha}, E_{M3}, H_{M\alpha}, H_{M3}, D_{\mathrm{R}\alpha}, D_{\mathrm{R}3}, D_{\mathrm{R}\alpha\beta}, D_{\mathrm{R}3\alpha}, A_{\alpha\beta}, z)$$
$$= \hat{\psi}^*(a_{\alpha\beta}, (-1)^R d_{\mathrm{R}\alpha}, (-1)^{R+1} d_{\mathrm{R}3}, (-1)^R d_{\mathrm{R}\alpha\beta}, (-1)^{R+1} d_{\mathrm{R}3\alpha}, \theta, (-1)^S \theta_S, (-1)^M E_{M\alpha},$$
$$(-1)^{M+1} E_{M3}, (-1)^{M+1} H_{M\alpha}, (-1)^M H_{M3}, (-1)^R D_{\mathrm{R}\alpha}, (-1)^{R+1} D_{\mathrm{R}3}, (-1)^R D_{\mathrm{R}\alpha\beta},$$
$$(-1)^{R+1} D_{\mathrm{R}3\alpha}, A_{\alpha\beta}, -z)$$
$$= \hat{\psi}^*(a_{\alpha\beta}, -d_{\mathrm{R}\alpha}, d_{\mathrm{R}3}, d_{\mathrm{R}\alpha\beta}, -d_{\mathrm{R}3\alpha}, \theta, \theta_S, E_{M\alpha}, -E_{M3}, -H_{M\alpha}, H_{M3}, -D_{\mathrm{R}\alpha}, D_{\mathrm{R}3}, D_{\mathrm{R}\alpha\beta},$$
$$-D_{\mathrm{R}3\alpha}, A_{\alpha\beta}, z) \quad \text{(C 13)}$$

if we recall that $\boldsymbol{H}_M$ is an axial vector. Similarly, using (C 5) and (C 9) we see that

case (*b*)

$$\hat{\psi}^*(a_{\alpha\beta}, d_{\mathrm{R}\alpha}, d_{\mathrm{R}3}, d_{\mathrm{R}\alpha\beta}, d_{\mathrm{R}3\alpha}, \theta, \theta_S, E_{M\alpha}, E_{M3}, H_{M\alpha}, H_{M3}, D_{\mathrm{R}\alpha}, D_{\mathrm{R}3}, D_{\mathrm{R}\alpha\beta}, D_{\mathrm{R}3\alpha}, A_{\alpha\beta}, z)$$
$$= \hat{\psi}^*(a_{\alpha\beta}, -d_{\mathrm{R}\alpha}, d_{\mathrm{R}3}, d_{\mathrm{R}\alpha\beta}, -d_{\mathrm{R}3\alpha}, \theta, \theta_S, E_{M\alpha}, -E_{M3}, -H_{M\alpha}, H_{M3}, -D_{\mathrm{R}\alpha}, D_{\mathrm{R}3}, D_{\mathrm{R}\alpha\beta},$$
$$-D_{\mathrm{R}3\alpha}, A_{\alpha\beta}, z). \quad \text{(C 14)}$$

Appendix D

We record here some results appropriate for the linear three-dimensional theory of a magnetic, polarized thermoelastic solid, which will be helpful in identification of constitutive coefficients in the direct formulation of the theory of plates. We suppose that the elastic solid in its reference state is homogeneous, at constant temperature $\overline{\theta}$, is unstressed and is free from electromagnetic fields; but is, in general, anisotropic. We use rectangular Cartesian coordinate axes x_i along a constant orthonormal system of base vectors $\boldsymbol{e}_i$, as well as vector and Cartesian tensor notation, throughout this Appendix. In the linearized theory, $\overline{\theta}+\theta^*$ is temperature, ρ^* is reference density, $\boldsymbol{u}^* = u_i^* \boldsymbol{e}_i$ is displacement, $e_{ij}^* = \frac{1}{2}(u_{i,j}^* + u_{j,i}^*)$ is strain, t_{ij} is the symmetric stress tensor, η^* is entropy density, ψ^* is free energy density, ξ^* is the internal rate of production of entropy density, $\boldsymbol{p}^* = p_i^* \boldsymbol{e}_i$ is the entropy flux vector and $\boldsymbol{E} = E_i \boldsymbol{e}_i$, $\boldsymbol{H} = H_i \boldsymbol{e}_i$, $\overline{\boldsymbol{D}} = \overline{D}_i \boldsymbol{e}_i$, $\boldsymbol{B} = B_i \boldsymbol{e}_i$, $\boldsymbol{J} = J_i \boldsymbol{e}_i$, E are the electromagnetic variables. The constitutive relations are:

$$\left.\begin{aligned}
\rho^*\psi^* &= \tfrac{1}{2}c_{ijrs}e_{ij}^* e_{rs}^* - c_{ij}e_{ij}^*\theta^* - \tfrac{1}{2}c\theta^{*2} - \tfrac{1}{2}f_{rs}E_r E_s - \tfrac{1}{2}g_{rs}H_r H_s \\
&\quad - h_{rs}E_r H_s - k_{rst}e_{rs}^* E_t - l_{rst}e_{rs}^* H_t + f_r E_r \theta^* + g_r H_r \theta^*, \\
t_{ij} &= c_{ijrs}e_{rs}^* - c_{ij}\theta^* - k_{ijt}E_t - l_{ijt}H_t, \\
\rho^*\eta^* &= c_{ij}e_{ij}^* - f_r E_r - g_r H_r + c\theta^*, \\
\overline{D}_r &= f_{rs}E_s + h_{rs}H_s + k_{ijr}e_{ij}^* - f_r\theta^*, \\
B_r &= g_{rs}H_s + h_{sr}E_s + l_{ijr}e_{ij}^* - g_r\theta^*, \\
p_i^* &= -k_{ij}\theta_{,j}^* - \bar{a}_{ij}E_j, \quad J_i = l_{ij}\theta_{,j}^* + b_{ij}E_j,
\end{aligned}\right\} \qquad \text{(D 1)}$$

$$J_i E_i - p_i^*\theta_{,i}^* = \rho^*(\overline{\theta}+\theta^*)\,\xi^* \geqslant 0. \qquad \text{(D 2)}$$

The condition (D 2) arises from thermodynamical considerations. The notation $(\ )_{,i}$ denotes partial differentiation with respect to x_i and the Cartesian tensor summation convention with repeated suffices is used. The various coefficients are constants and subject to the following restrictions:

$$c_{ijrs} = c_{jirs} = c_{ijsr} = c_{rsij}, \quad c_{ij} = c_{ji}, \quad f_{rs} = f_{sr}, \quad g_{rs} = g_{sr}, \quad k_{rst} = k_{srt}, \quad l_{rst} = l_{srt}. \qquad \text{(D 3)}$$

In making use of these results it is convenient to express them in a partially inverted form

$$\left.\begin{aligned}
e_{ij}^* &= s_{ijrs}t_{rs} + s_{ij}\theta^* - k_{ijt}^* E_t - l_{ijt}^* H_t, \\
\rho^*\eta^* &= s_{ij}t_{ij} - f_r^* E_r - g_r^* H_r + c^*\theta^*, \\
\overline{D}_r &= -k_{ijr}^* t_{ij} - f_r^*\theta^* - f_{rs}^* E_s - h_{rs}^* H_s, \\
B_r &= -l_{ijr}^* t_{ij} - g_r^*\theta^* - h_{sr}^* E_s - g_{rs}^* H_s,
\end{aligned}\right\} \qquad \text{(D 4)}$$

where the coefficients have symmetry restrictions similar to the corresponding coefficients in (D 3). Also

$$\left.\begin{aligned}
&c_{ijrs}s_{rsmn} = \tfrac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \quad c_{ijrs}s_{rs} - c_{ij} = 0, \quad c - c^* + c_{ij}s_{ij} = 0, \\
&c_{ijrs}k_{rst}^* + k_{ijt} = 0, \quad c_{ijrs}l_{rst}^* + l_{ijt} = 0, \quad f_r^* = f_r - s_{ij}k_{ijr}, \quad g_r^* = g_r - s_{ij}l_{ijr}, \\
&f_{rs}^* + f_{rs} = k_{ijr}k_{ijs}^*, \quad h_{rs}^* + h_{rs} = k_{ijr}l_{ijs}^*, \quad g_{rs}^* + g_{rs} = l_{ijr}l_{ijs}^*,
\end{aligned}\right\} \qquad \text{(D 5)}$$

with similar formulae in which c_{ijrs}, c_{ij} are interchanged with s_{ijrs}, s_{ij}, respectively, and in which starred and unstarred quantities are interchanged.

Material symmetries will restrict the number of independent coefficients in (D 1). Here we list the restrictions imposed on (D 1) for two cases. If the material is orthotropic with respect to the orthonormal system of vectors $\boldsymbol{e}_i$, then the only non-zero coefficients in (D 1) and (D 2) are

$$\left.\begin{array}{l} c_{1111}, c_{1122}, c_{2222}, c_{1212}, c_{1133}, c_{2233}, c_{1313}, \\ c_{2323}, c_{3333}, c_{11}, c_{22}, c_{33}, c, f_{11}, f_{22}, f_{33}, \\ g_{11}, g_{22}, g_{33}, l_{123}, l_{231}, l_{312}, \\ k_{11}, k_{22}, k_{33}, b_{11}, b_{22}, b_{33}, l_{11}, l_{22}, l_{33}, \bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33} \end{array}\right\} \qquad \text{(D 6)}$$

together with the coefficients connected with (D 6) by the relations (D 3).

The second class of materials, which includes as a special case the rotated Y-cut quartz plate, is such that there is symmetry with respect to rotation about the $\boldsymbol{e}_2$ direction through an angle π. For this class the only non-zero coefficients in (D 1) and (D 2) are

$$\left.\begin{array}{l} c_{1111}, c_{1122}, c_{1133}, c_{2222}, c_{2233}, c_{3333}, c_{1212}, c_{1313}, c_{2323}, c_{1113}, c_{2213}, c_{3313}, c_{1223}, \\ c_{11}, c_{22}, c_{33}, c_{13}, c, f_2, g_2, f_{11}, f_{22}, f_{33}, f_{13}, g_{11}, g_{22}, g_{33}, g_{13}, \\ h_{11}, h_{13}, h_{31}, h_{33}, h_{22}, k_{121}, k_{123}, k_{231}, k_{233}, k_{112}, k_{222}, k_{132}, k_{332}, \\ l_{121}, l_{123}, l_{231}, l_{233}, l_{112}, l_{222}, l_{132}, l_{332}, k_{11}, k_{22}, k_{33}, k_{13}, b_{11}, b_{22}, b_{33}, b_{13}, \\ \bar{a}_{11}, \bar{a}_{22}, \bar{a}_{33}, \bar{a}_{13}, l_{11}, l_{22}, l_{33}, l_{13}, \end{array}\right\} \qquad \text{(D 7)}$$

together with coefficients connected with (D 7) by the relations (D 3).

For some purposes we need to express the Helmholtz free energy in terms of functions defined in a general curvilinear system of coordinates. Then, (D 1) is replaced by

$$\begin{aligned} \rho^*\psi^* = {} & \tfrac{1}{2}c^{ijrs}e^*_{ij}\,e^*_{rs} - c^{ij}e^*_{ij}\,\theta^* - \tfrac{1}{2}c\theta^{*2} - \tfrac{1}{2}f^{rs}E_rE_s - \tfrac{1}{2}g^{rs}H_rH_s - h^{rs}E_rH_s \\ & - k^{rst}e_{rs}E_t - l^{rst}e_{rs}H_t + f^rE_r\theta^* + g^rH_r\theta^*, \end{aligned} \qquad \text{(D 8)}$$

where the coefficients may now depend on the curvilinear coordinates x^i even if the material is homogeneous. The constitutive equations for p^{*i}, J^i become

$$p^{*i} = -k^{ij}\theta^*_{,j} - \bar{a}^{ij}E_j, \quad J^i = l^{ij}\theta^*_{,j} + b^{ij}E_j. \qquad \text{(D 9)}$$

References

Bugdayci, N. & Bogy, B. B. 1981 *Int. J. Solids Struct.* **17**, 1159 and 1179.
Green, A. E. & Naghdi, P. M. 1976 *Proc. R. Soc. Lond.* A **347**, 447.
Green, A. E. & Naghdi, P. M. 1977 *Proc. R. Soc. Lond.* A **357**, 253.
Green, A. E. & Naghdi, P. M. 1979 *Proc. R. Soc. Lond.* A **365**, 161.
Green, A. E. & Naghdi, P. M. 1982 *IMA J. appl. Math.* **29**, 1–23.
Green, A. E., Naghdi, P. M. & Wainwright, W. L. 1965 *Arch. ration. Mech. Analysis* **20**, 287.
Hutter, K. & van de Ven, A. A. F. 1978 *Field matter interactions in thermoelastic solids. Lecture notes in physics*, no. 88, Berlin: Springer-Verlag.
Mindlin, R. D. 1961 *Q. appl. Math.* **19**, 51.
Naghdi, P. M. 1972 The theory of shells and plates. *Flügge's Handbuch der Physik*, vol. VI a/2, p. 425. Berlin, Heidelberg, New York: Springer-Verlag.
Naghdi, P. M. 1977 Shell theory from the standpoint of finite elasticity. In *Proc. Symp. on Finite Elasticity* (ed. R. S. Rivlin), AMD-Vol. 27, p. 77. New York: Am. Soc. Mech. Engrs.
Naghdi, P. M. 1982 Finite deformation of elastic rods and shells. In *Proc. IUTAM Symp. on Finite Elasticity* (ed. D. E. Carlson and R. T. Shield), p. 47. The Hague, The Netherlands: Martinus Nijhoff.
Tiersten, H. F. 1969 *Linear piezo-electric plate vibrations*. New York: Plenum Press.
Tiersten, H. F. & Mindlin, R. D. 1962 *Q. appl. Math.* **20**, 107.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES